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Investigations in the Lunar Theory.

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This paper is an outline of a plan for the development of the expressions which represent the coordinates of the Moon, together with certain theorems connected with the infinite determinants which determine the motions of Perigee and Node, and, in addition, some results concerning the constant part of the expression which gives the Parallax of the Moon.

It has been pointed out that the algebraical expressions of the coordinates show slow convergence chiefly when the coefficients of the periodic terms are arranged in powers of the ratio of the mean motions of the Sun and Moon, and that when we use the numerical value of this ratio from the start, keeping the other constants arbitrary, slow convergence in the series arranged in powers and products of the other constants, is unusual; also, the observed value of the ratio of the mean motions is determined with an accuracy far surpassing that with which the other constants are known. These reasons have seemed to point towards a semi-algebraical development, in which the ratio of the mean motions is given its numerical value while the other constants are left arbitrary.

The developments given below are built up on this basis. The orbit of Dr. Hill, which depends on m only, is, therefore, a numerical one, and is used as a first approximation or "intermediate" orbit.

But there are several methods by which we might proceed with the development from this point. It is possible to use polar or rectangular coordinates, and the equations of motion in each case can be given several different forms. Of the forms which the equations may take when rectangular coordinates are used, there are two which seem specially adapted for continued approximation. Of these one set are of the second degree and second order when we neglect the Parallax of the Sun and the Latitude of the Moon. When these quantities are included, two equations still keep a form in which the principal terms are of the same kind, while the third equation is a sufficiently simple one. These are the equations (8) below. Some of the elliptic and parallactic terms have been determined from these equations.

Another method is to use the equations of motion in their original form. This requires the expansion of such expressions as $(x_0 + \delta x)/(r_0 + \delta r)^3$, where x_0, r_0 represent the values of x, r in the intermediate orbit. These being trigonometrical or exponential series, entail a certain amount of labor in the developments. How this is effected and in what way we gain or lose in comparison to the former method is shown with some detail in Part I. It is on these lines that the developments there given are carried out.

The chief difficulties which arise in any method of treating the Lunar Theory are those connected with the determinations of the motions of the Perigee and Node. When Hill and Adams conceived the idea of the infinite determinant, and succeeded in solving one, there appeared to be an opening for the determination of these motions with an accuracy which should leave nothing to be desired. The equation for the latitude in rectangular coordinates is in a form which gives the principal part of the motion of the Node—that depending on m only—by means of an infinite determinant directly, and thence the coefficients of the periodic terms are easily obtained. But in the case of the Perigee this is not so. Several transformations are necessary before a suitable determinant is evolved, and when these have been made and the coefficients of the transformed solution obtained, the return process to find the coefficients of the periodic terms in rectangular coordinates is a laborious one. Once the motion of the Perigee has been obtained, however, we can use the equations in any form we choose.

So far the question, then, of finding the principal parts of the motion of Perigee and Node is resolved, and the coefficients dependent on these can be obtained. But we are again met by the same difficulties when we try to find further approximations to these motions. We again obtain an infinite set of linear equations. But they now contain constant terms, and though an infinite determinant can be obtained for the new part of the Perigee or Node to be found, it is in a form which does not admit of easy solution. The equations can be solved by continued approximation, but this is very laborious. The difficulty has been turned by a slight artifice which eliminates all the unknown quantities from the set of equations but that required, and the resulting equation is one which does not present any unusual difficulties in calculation.

The equations thus obtained not only showed methods by which the motions of Perigee and Node can be obtained at any stage by simple processes, but indicate the way in which Adams' theorems connecting these motions with the constant part of the expression of the Parallax arise. The remarkable proof Adams gives of his theorems is well adapted to the end in view, but it shows no indication of any possible extension to the higher terms. The methods given below, though cumbrous, have perhaps the advantage of shedding a clearer light on the theorems and of giving them a form which admits of further development. The investigation of these occupies Part II. Other theorems are also proved which may be of value when a verification of results is required.

Part III contains one or two deductions from the infinite determinant concerning the orders of the coefficients, and especially deals with the cases in which short-period terms may have large coefficients.

For convenience, the papers referred to in the following pages—

On the Lunar Theory:

I. G. W. Hill.—Researches in the Lunar Theory. American Journal of Mathematics, Vol. I, pp. 5–26, 129–147, 245–260.

II. G. W. Hill.—On the Part of the Motion of the Lunar Perigee, etc. Acta Mathematica, Vol. VIII, pp. 1–35.

III. J. C. Adams.—On the Motion of the Moon's Node in the case, etc. Monthly Notices R. A. S., Vol. XXXVIII, pp. 43–49.

IV. J. C. Adams.—Note on a Remarkable Property of the Analytical Expression, etc. Monthly Notices R. A. S., Vol. XXXVIII, pp. 460–472.

V. C. Delaunay.—Théorie de la Lune. Mémoires de l'Academie des Sciences, Vols. XXVIII, XXIX.

VI. C. Delaunay.—Note sur les mouvements du périgée et du nœud de la Lune. Comptes Rendus, Vol. LXXIV, pp. 17–21.

VII. E. W. Brown.—The Parallactic Inequalities in the Lunar Theory. American Jour. Math., Vol. XIV, pp. 141–160.

VIII. E. W. Brown.—The Elliptic Inequalities in the Lunar Theory. Amer. Jour. Math., Vol. XV, pp. 243–263, 321–336.

And on the Infinite Determinant:

IX. H. Poincaré.—Sur les déterminants d'ordre infini. Bulletin de la Société math. de France, Vol. XIV, pp. 77–90.

X. Helge von Kock.—Sur les determinants infinis, etc. Acta Math., Vol. XVI, pp. 217–295.

—are gathered together and the reference is made by means of the number in Roman numerals standing before each.

The portion of I dealing with the intermediate orbit is also found in Tisserand's Mécanique Céleste, Vol. III, Chapter XIV. The Memoir II is given in Chapter XV, and III and IV in Chapter XVI of the same volume.

PART I.

DEVELOPMENT OF THE THEORY.

1. *The Differential Equations.*

The notations to be adopted below are as follows:

n, n', the mean angular motions of the Sun and Moon about the Earth;
$2a'$, r', e', the major axis, radius vector and eccentricity of the Sun's orbit, supposed elliptic;
x, y, z, the coordinates of the Moon referred to rectangular axes, of which those of x, y are moving in the plane of the Sun's orbit with angular velocity n', and that of z is perpendicular to this plane. The positive direction of the x-axis is directed to the mean place of the Sun;
v, the solar equation of the centre;
$S = x \cos v + y \sin v$;
$u = x + y\sqrt{-1}$, $s = x - y\sqrt{-1}$, $r^2 = us = x^2 + y^2$;
$n'^2 a'^3$, the mass of the Sun;
μ, the sum of the masses of the Earth and Moon in the same units;
$\nu = n - n'$, $m = n/\nu$, $\varkappa = \mu/\nu^2$;

$$\zeta = e^{\nu(t-t_0)\sqrt{-1}}, \quad D = \zeta \frac{d}{d\zeta} = \frac{1}{\nu\sqrt{-1}} \frac{d}{dt}.$$

With the limitations here adopted—the same as those considered by Delaunay (V) and Hill (I)—the potential function is

$$\Omega = \frac{\mu}{\sqrt{r^2 + z^2}} + \frac{n'^2 a'^3}{\sqrt{[r'^2 - 2Sr' + r^2 + z^2]}} - \frac{n'^2 a'^3}{r'^2} S,$$

and the equations of motion are

$$\begin{aligned} \ddot{x} - 2n'\dot{y} - n'^2 x &= \frac{\partial \Omega}{\partial x}, \\ \ddot{y} + 2n'\dot{x} - n'^2 y &= \frac{\partial \Omega}{\partial y}, \\ \ddot{z} &= \frac{\partial \Omega}{\partial z}. \end{aligned}$$

If

$$\Omega' = \Omega + \tfrac{1}{2} n'^2 (x^2 + y^2) = \Omega + \tfrac{1}{2} n'^2 r^2,$$

the equations may be written

$$\left.\begin{aligned} \ddot{x} - 2n'\dot{y} &= \frac{\partial \Omega'}{\partial x}, \\ \ddot{y} + 2n'\dot{x} &= \frac{\partial \Omega'}{\partial y}, \\ \ddot{z} &= \frac{\partial \Omega'}{\partial z}. \end{aligned}\right\} \qquad (1)$$

Transforming to the independent variables u, s, z and the dependent ζ, the equations become

$$\begin{aligned} D^2 u + 2mDu &= -\frac{2}{\nu^2} \cdot \frac{\partial \Omega'}{\partial s}, \\ D^2 s - 2mDs &= -\frac{2}{\nu^2} \cdot \frac{\partial \Omega'}{\partial u}, \\ D^2 z &= -\frac{1}{\nu^2} \cdot \frac{\partial \Omega'}{\partial z}, \end{aligned}$$

with the expression

$$\Omega' = \frac{\mu}{\sqrt{us + z^2}} + \frac{n'^2 a'^3}{\sqrt{r'^2 - 2Sr' + us + z^2}} - \frac{n'^2 a'^3}{r'^2} S + \tfrac{1}{2} n'^2 us.$$

This becomes by expansion

$$\begin{aligned} \frac{2}{\nu^2} \Omega' &= \frac{2\mu}{\nu^2 \sqrt{us + z^2}} + m^2 us + m^2 \frac{a'^3}{r'^3} [3S^2 - us - z^2] \\ &\qquad + \frac{m^2}{a'} \cdot \frac{a'^4}{r'^4} [5S^3 - 3S(us + z^2)] + \ldots . \\ &= \frac{2\varkappa}{\sqrt{us + z^2}} + \tfrac{3}{4} m^2 (u + s)^2 - m^2 z^2 + \Omega_1, \end{aligned}$$

where

$$\begin{aligned} \Omega_1 = 3m^2 \left[\frac{a'^3}{r'^3} S^2 - \tfrac{1}{4} (u + s)^2 \right] &- m^2 (us + z^2) \left(\frac{a'^3}{r'^3} - 1 \right) \\ &+ \frac{m^2}{a'} \cdot \frac{a'^4}{r'^4} [5S^3 - 3S(us + z^2)] + \ldots . \qquad (2) \end{aligned}$$

It is necessary, before going further, to see in what way Ω_1 involves u, s. We have

$$\begin{aligned} S^2 &= (x \cos v + y \sin v)^2 \\ &= \tfrac{1}{4}(u + s)^2 - \tfrac{1}{2}(u^2 + s^2)\sin^2 v + \frac{1}{4\sqrt{-1}}(u^2 - s^2)\sin 2v, \end{aligned}$$

and if

$$\frac{a'^3}{r'^3} = 1 + \rho,$$

the first two terms of Ω_1 become

$$3m^2\left[\frac{\rho}{4}(u + s)^2 - \frac{1 + \rho}{2}(u^2 + s^2)\sin^2 v + \frac{1 + \rho}{4\sqrt{-1}}(u^2 - s^2)\sin 2v\right] - \rho m^2(us + z^2).$$

From the known properties of elliptic motion ρ, $\sin v$ are of the order e' at least. Hence Ω_1 is of the order e' or $1/a'$ at least. When we neglect $1/a'$, Ω_1 takes the form

$$\Omega_1 = \tfrac{1}{2}(Au^2 + 2Bus + Cs^2 - 2\rho m^2 z^2), \tag{3}$$

where A, B, C depend on e' and the angle $n't + \epsilon'$ only. When $1/a'$ is not neglected we can put

$$\Omega_1 = \omega_2 + \omega_3 + \ldots\ldots, \tag{3'}$$

where ω_q is a homogeneous function of u, s, z, of degree q free from fractions. Also since the time enters explicitly into Ω_1 only through the coordinates of the Sun, when $e' = 0$, Ω_1 does not contain the time explicitly.

In the general case the equations may now be written

$$\left.\begin{aligned} (D^2 + 2mD + \tfrac{3}{2}m^2)u + \tfrac{3}{2}m^2 s - \frac{\varkappa u}{(us + z^2)^{\frac{3}{2}}} &= -\frac{\partial\Omega_1}{\partial s}, \\ (D^2 - 2mD + \tfrac{3}{2}m^2)s + \tfrac{3}{2}m^2 u - \frac{\varkappa s}{(us + z^2)^{\frac{3}{2}}} &= -\frac{\partial\Omega_1}{\partial u}, \\ (D^2 - m^2)z \qquad\qquad\qquad - \frac{\varkappa z}{(us + z^2)^{\frac{3}{2}}} &= -\tfrac{1}{2}\frac{\partial\Omega_1}{\partial z}, \end{aligned}\right\} \tag{4}$$

which are the equations to be used in developing this theory.

Taking the general value of Ω_1, multiply the first of these equations by s, the second by u and subtract, we obtain

$$D(uDs - sDu - 2mus) + \tfrac{3}{2}m^2(u^2 - s^2) = s\frac{\partial\Omega_1}{\partial s} - u\frac{\partial\Omega_1}{\partial u}. \tag{5}$$

With the same multipliers for the first two equations and the multiplier $2z$ for the last, we have by addition

$$\left.\begin{aligned} sD^2u + uD^2s + 2zD^2z + 2m\,(sDu - uDs) + \tfrac{3}{2}\,m^2\,(u+s)^2 - 2m^2z^2 & \\ - \frac{2\kappa}{(us+z^2)^{\frac{1}{2}}} = - \left(s\,\frac{\partial\Omega_1}{\partial s} + u\,\frac{\partial\Omega_1}{\partial u} + z\,\frac{\partial\Omega_1}{\partial z} \right) & \\ = -\sum_{q=2}^{\infty} q\omega_q \quad \text{by } (3'). & \end{aligned}\right\} \qquad (6)$$

Again, multiplying the three equations by Ds, Du, $2Dz$ respectively and adding, we obtain

$$D\left[Du.Ds + (Dz)^2 + \tfrac{9}{4}\,m^2\,(u+s)^2 - m^2z^2 + \frac{2\kappa}{(us+z^2)^{\frac{1}{2}}}\right] = -\left[\frac{\partial\Omega_1}{\partial s}\,Ds + \frac{\partial\Omega_1}{\partial u}\,Du + \frac{\partial\Omega_1}{\partial z}\,Dz\right].$$

Now in the general case Ω_1 is expressed explicitly in terms of u, s, z, t, hence

$$D\Omega_1 = \frac{\partial\Omega_1}{\partial u}\,Du + \frac{\partial\Omega_1}{\partial s}\,Ds + \frac{\partial\Omega_1}{\partial z}\,Dz + \frac{\partial\Omega_1}{\partial t}\,Dt,$$

and therefore the right-hand side of the previous equation is

$$\begin{aligned} \frac{\partial\Omega_1}{\partial t}\,Dt - D\Omega_1 &= \frac{1}{\nu\sqrt{-1}}\cdot\frac{\partial\Omega_1}{\partial t} - D\Omega_1 \\ &= D'\Omega_1 - D\Omega_1 \\ &= D\,[D^{-1}(D'\Omega_1) - \Omega_1], \end{aligned}$$

where $D'\Omega_1$ denotes the operation D performed on Ω_1 only with reference to t (or ζ) so far as it occurs explicitly in Ω_1, and D^{-1} denotes the operation inverse to D, i. e. integration with respect to ζ followed by a division by ζ.

If, then, with these substitutions we integrate, the equation becomes

$$Du.Ds + (Dz)^2 + \tfrac{9}{4}\,m^2\,(u+s)^2 - m^2z^2 + \frac{2\kappa}{(us+z^2)^{\frac{1}{2}}} = C' - \Omega_1 + D^{-1}(D'\Omega_1). \qquad (7)$$

Adding this to (6) and writing down again equations (5) and the last of equations (4), the three transformed equations of motion are

$$\left.\begin{aligned} D^2(us+z^2) - Du.Ds - (Dz)^2 - 2m\,(uDs - sDu) + \tfrac{9}{4}\,m^2\,(u+s)^2 - 3m^2z^2 & \\ = C' - \sum_{q=2}^{\infty}(q+1)\,\omega_q + D^{-1}\,(D'\Omega_1), & \\ D\,(uDs - s\,Du - 2mus) + \tfrac{3}{2}\,m^2\,(u^2 - s^2) = s\,\frac{\partial\Omega_1}{\partial s} - u\,\frac{\partial\Omega_1}{\partial u}, & \\ D^2z - m^2z - \frac{\kappa z}{(us+z^2)^{\frac{3}{2}}} = -\tfrac{1}{2}\,\frac{\partial\Omega_1}{\partial z}. & \end{aligned}\right\} \qquad (8)$$

There are some special cases to be noticed:

(i) $e' = 0$, the last term of the first equation disappears and $\omega_2 = 0$;

(ii) $\dfrac{1}{a'} = 0$, then $\omega_q = 0$, except for the value $q = 2$, $\Omega_1 = \omega_2$;

(iii) $z = 0$, the third equation disappears;

(iv) $e' = 0$, $\dfrac{1}{a'} = 0$, all the terms on the right-hand sides disappear except the constant C';

(v) $e' = 0$, $1/a' = 0$, $z = 0$, the equations reduce to those studied by Dr. Hill (I, II) and they give those inequalities which depend on e, m only.

We are going chiefly to study the equations of motion under the form (4), and it will be advisable to write down the special forms to which they reduce for zero values of certain of the constants. In the majority of the cases to be treated $1/a'$ is neglected, so that Ω_1 has the value (3) and the equations become

$$\left.\begin{aligned}(D^2 + 2mD + \tfrac{3}{2}m^2)\,u + \tfrac{3}{2}m^2 s + Bu + Cs - \frac{\varkappa u}{(us + z^2)^{\frac{3}{2}}} &= 0,\\ (D^2 - m^2)\,z - \rho m^2 z - \frac{\varkappa z}{(us + z^2)^{\frac{3}{2}}} &= 0,\end{aligned}\right\} \qquad (9)$$

(It is not necessary to write down the s-equation, since u is a complex quantity and the u-equation includes both the x-equation and the y-equation.) The particular cases of (9) are

$$z = 0, \qquad (D^2 + 2mD + \tfrac{3}{2}m^2)\,u + \tfrac{3}{2}m^2 s + Bu + Cs - \frac{\varkappa u}{(us)^{\frac{3}{2}}} = 0, \qquad 9\,\text{(i)}$$

$$z = 0,\ e' = 0, \qquad B = 0,\quad C = 0,\ \rho = 0 \text{ in equation } 9\,\text{(i)}, \qquad 9\,\text{(ii)}$$

$$e' = 0, \qquad B = 0,\quad C = 0,\ \rho = 0 \text{ in equations (9)}. \qquad 9\,\text{(iii)}$$

We shall require the equations (9) expressed in terms of x, y, z. They are

$$\left.\begin{aligned}\ddot{x} - 2n'\dot{y} - 3n'^2 x + A'x + B'y &= -\frac{\mu x}{(r^2 + z^2)^{\frac{3}{2}}},\\ \ddot{y} + 2n'\dot{x} \qquad\quad + B'x + C'y &= -\frac{\mu y}{(r^2 + z^2)^{\frac{3}{2}}},\\ \ddot{z} + n'^2 z + n'^2 \rho z \qquad\qquad &= -\frac{\mu z}{(r^2 + z^2)^{\frac{3}{2}}},\end{aligned}\right\} \qquad (10)$$

where A', B', C', ρ are quantities depending on e' and arising from the part ω_2 of the disturbing function.

2. *Method of Development.*

We take first the equation 9 (ii). It is known (Hill, I) that in this case the equations admit of a solution which can be expressed in the form

$$\left.\begin{aligned} u &= u_0 = a_0 \Sigma a_i \zeta^{2i+1} \\ s &= s_0 = a_0 \Sigma a_{-i-1} \zeta^{2i+1} \end{aligned}\right\} \qquad i = -\infty \ldots\ldots +\infty, \tag{11}$$

where a_i depends on m only. This is Dr. Hill's primary solution or intermediate orbit. It contains only two arbitrary constants, while the general solution of the equation 9 (ii) contains four. It is desired to know the general solution of this equation and also of equations 9, 9 (i), 9 (iii). We shall for simplicity at present take the equation 9 (i), the extension so as to include the terms dependent on z being perfectly evident. We may write it

$$(D+m)^2 u + \tfrac{1}{2} m^2 u + \tfrac{3}{2} m^2 s - \frac{\varkappa u}{(us)^{\frac{3}{2}}} = - Bu - Cs,$$

where the terms on the right-hand side are of the order e' at the lowest. If now we put

$$\begin{aligned} u &= u_0 + \delta u, \\ s &= s_0 + \delta s\,, \end{aligned}$$

u_0, s_0 satisfy the equation

$$(D+m)^2 u_0 + \tfrac{1}{2} m^2 u_0 + \tfrac{3}{2} m^2 s_0 - \frac{\varkappa u_0}{(u_0 s_0)^{\frac{3}{2}}} = 0,$$

and δu, δs will therefore be determined by means of the equation

$$\begin{aligned} (D+m)^2 \delta u + \tfrac{1}{2} m^2 \delta u + \tfrac{3}{2} m^2 \delta s - \left[\frac{\varkappa}{(u_0+\delta u)^{\frac{1}{2}} (s_0+\delta s)^{\frac{3}{2}}} - \frac{\varkappa}{u_0^{\frac{1}{2}} s_0^{\frac{3}{2}}}\right] \\ = - B(u_0 + \delta u) - C(s_0 + \delta s). \quad (12) \end{aligned}$$

We now suppose that δu, δs are small enough for the expression in square brackets to be expanded in powers and products of these quantities.

The expansion of this portion is

$$\frac{\varkappa}{u_0^{\frac{1}{2}} s_0^{\frac{3}{2}}}\Big[-\frac{1}{2}\,\frac{\delta u}{u_0}-\frac{3}{2}\,\frac{\delta s}{s_0}+\frac{3}{8}\,\frac{(\delta u)^2}{u_0^2}+\frac{15}{8}\,\frac{(\delta s)^2}{s_0^2}$$
$$+\frac{3}{4}\,\frac{\delta u\,\delta s}{u_0 s_0}-\frac{5}{16}\,\frac{(\delta u)^3}{u_0^3}-\frac{35}{16}\,\frac{(\delta s)^3}{s_0^3}-\frac{9}{16}\,\frac{(\delta u)^2\,\delta s}{u_0^2 s_0}-\frac{15}{16}\,\frac{\delta u\,(\delta s)^2}{u_0 s_0^2}+\ldots\ldots\Big]$$

As u_0, s_0 are supposed known, we can, from their values, obtain the coefficients of the various powers of δu, δs in the form of known series. It is understood that the numerical value of m is always used, so that these series can all be expressed as odd *or* even power series in ζ with numerical coefficients. Since

$$u_0\zeta^{-1}=a_0\Sigma a_i\zeta^{2i}\ ,$$
$$s_0\zeta\ \ =a_0\Sigma a_{-i}\zeta^{2i},$$

all the series can be expressed in terms of even positive and negative powers of ζ, so that, by a suitable arrangement, the largest coefficient is that of ζ^0. It is assumed, unless otherwise stated, that the summations include all positive and negative integral values of i, including zero.

Let

$$\tfrac{1}{2}m^2+\tfrac{1}{2}\frac{\varkappa}{(u_0 s_0)^{\frac{3}{2}}}=\Sigma M_i\zeta^{2i}\ ,\qquad \tfrac{3}{2}m^2\zeta^{-2}+\tfrac{3}{2}\frac{\varkappa\zeta^{-2}}{u_0^{\frac{1}{2}}s_0^{\frac{5}{2}}}=\Sigma N_i\zeta^{2i},$$
$$\frac{\varkappa\zeta}{u_0^{\frac{5}{2}}s_0^{\frac{3}{2}}}=\Sigma P_{-i}\zeta^{2i},\quad \frac{\varkappa\zeta^{-1}}{u_0^{\frac{3}{2}}s_0^{\frac{5}{2}}}=\Sigma P_i\zeta^{2i}\ ,\quad \frac{\varkappa\zeta^{-3}}{u_0^{\frac{1}{2}}s_0^{\frac{7}{2}}}=\Sigma Q_i\zeta^{2i},$$
$$\frac{\varkappa\zeta^2}{u_0^{\frac{7}{2}}s_0^{\frac{3}{2}}}=\Sigma R_i\zeta^{2i}\ ,\quad \frac{\varkappa\zeta^{-4}}{u_0^{\frac{1}{2}}s_0^{\frac{9}{2}}}=\Sigma T_i\zeta^{2i}\ ,$$
$$\frac{\varkappa}{u_0^{\frac{5}{2}}s_0^{\frac{5}{2}}}=\Sigma S_i\zeta^{2i}\ ,\quad \frac{\varkappa\zeta^{-2}}{u_0^{\frac{3}{2}}s_0^{\frac{7}{2}}}=\Sigma R_{-i}\zeta^{2i}.$$

When ζ^{-1} is put for ζ, u_0 changes to s_0 and s_0 to u_0. Hence $M_i=M_{-i}$ and $S_i=S_{-i}$.

As for the calculation of these, M_i is immediately obtained from the value of $\varkappa/r_0^3$ given by Dr. Hill (I, p. 249). For N_i we have

$$\frac{\varkappa\zeta^{-2}}{u_0^{\frac{1}{2}}s_0^{\frac{5}{2}}}=\frac{\varkappa}{r_0^3}\cdot\frac{u^2\zeta^{-2}}{r_0^2}\,.$$

We calculate $1/r_0^2$ and the real and imaginary parts of $u^2\zeta^{-2}$. These are easily done by the method of special values, which method will also give quickly the required coefficients. In the same way we may find the other coefficients P, Q, R.... The whole process does not entail very much labor.

The following table gives the values of M, N, P, Q:

$M_0 = +.58902\ 22856\ 4$ $M_{\pm3} = +.00000\ 06029\ 7$
$M_{\pm1} = +.00630\ 84231\ 2$ $M_{\pm4} = +.00000\ 00056\ 5$
$M_{\pm2} = +.00006\ 28883\ 4$ $M_{\pm5} = +.00000\ 00000\ 5$
$N_0 = +1.75707\ 88032\ 7$
$N_1 = +.03686\ 55171\ 8$ $N_{-1} = +.01078\ 63527\ 2$
$N_2 = +.00054\ 79401\ 6$ $N_{-2} = +.00001\ 25690\ 4$
$N_3 = +.00000\ 70129\ 7$ $N_{-3} = +.00000\ 00982\ 3$
$N_4 = +.00000\ 00824\ 6$ $N_{-4} = +.00000\ 00007\ 6$
$N_5 = +.00000\ 00009\ 0$
$P_0 = 1.17156\ 77322$
$P_1 = +.02280\ 40093$ $P_{-1} = +.01084\ 18484$
$P_2 = +.00032\ 38766$ $P_{-2} = +.00010\ 24640$
$P_3 = +.00000\ 40164$ $P_{-3} = +.00000\ 09526$
$P_4 = +.00000\ 00465$ $P_{-4} = +.00000\ 00092$
$P_5 = +.00000\ 00005$ $P_{-5} = +.00000\ 00001$
$Q_0 = 1.17132\ 34260$
$Q_1 = +.03476\ 15314$ $Q_{-1} = -.00112\ 12092$
$Q_2 = +.00066\ 73632$ $Q_{-2} = +.00000\ 31923$
$Q_3 = +.00001\ 04714$ $Q_{-3} = +.00000\ 00337$
$Q_4 = +.00000\ 01462$ $Q_{-4} = +.00000\ 00007$
$Q_5 = +.00000\ 00020$

The numerical values of these coefficients have been given here because they are fundamental for this method of development. Further, they are the same for every series of inequalities which it may be desired to calculate. The values of M, N, P, Q given above will suffice to determine all terms in u, s, depending on e, e', $1/a'$, γ^2 as far as the first and second powers and the products, two at a time, of these quantities are concerned, and in z of the same classes of inequalities each multiplied by γ. These form by far the larger portion of the expressions given by Delaunay.

When we wish to determine the terms dependent on a' there are certain parts of the disturbing function to be added to the right-hand sides of the equations. But as these are homogeneous functions of u, s of the second, third, ... orders, their expansion is quite easy, and in order to keep the exposition as simple as possible they have not been written down.

The various cases will now be considered in the order which seems most suitable to the objects in view, namely, the development of the theory and the special theorems to be treated in Part II.

3. *The Terms whose Coefficients depend on m, e' only.*

We take equation (12) and it is desired to find a particular solution of it such that the coefficients depend only on m, e'. As the primary solution depends only on m, and as the introduction of e' brings with it multiples of the angle $n't + \varepsilon'$, no new constant of integration will be introduced. Also δu, δs will here contain e' as a factor.

At the beginning we neglect powers of e', and therefore of δu, δs above the first, and put for these terms

$$\delta u = u_\eta, \quad \delta s = s_\eta,$$

and the equation may be written

$$\zeta^{-1}(D+m)^2 u_\eta + \zeta^{-1} u_\eta . \Sigma M_i \zeta^{2i} + \zeta s_\eta . \Sigma N_i \zeta^{2i} = -B\zeta^{-1}u_0 - C\zeta^{-2}.\zeta s_0 .$$

The coefficients B, C contain t only in the form $\frac{\sin}{\cos}(n't+\varepsilon')$ or in the form $\zeta^{\pm m}$. (The constant part of the angle may be omitted since we suppose that m is not commensurable with any whole number, and ε' plays no part in finding the coefficients of the terms in u_η, s_η.) The solution is therefore of the form

$$\begin{aligned}\zeta^{-1}u_\eta &= a_0 e' \Sigma_j \left[\eta_j \zeta^{2j+m} + \eta'_j \zeta^{2j-m}\right],\\ \zeta s_\eta &= a_0 e' \Sigma_j \left[\eta'_{-j}\zeta^{2j+m} + \eta_{-j}\zeta^{2j-m}\right],\end{aligned}$$

where j takes all positive and negative integral values including zero.

The equations which determine the coefficients η_j, η'_{-j} are

$$\begin{aligned}(2j+1+2m)^2\eta_j + \Sigma_i M_i \eta_{j-i} + \Sigma_i N_i \eta'_{i-j} &= \text{coeff. of } \zeta^{2j+m} \text{ in } -\Sigma_j\left[Ba_j + Ca_{-j-1}\right]\zeta^{2j},\\ (2j-1)^2\eta'_{-j} + \Sigma_i M_{-i}\eta'_{i-j} + \Sigma_i N_{-i}\eta_{j-i} &= \text{coeff. of } \zeta^{-2j-m} \text{ in } -\Sigma_j\left[Ba_j + Ca_{-j-1}\right]\zeta^{2j}.\end{aligned}$$

From these we can find η_j, η'_{-j} in terms of m by continued approximation.

The next step is to find the terms of order e'^2 by putting

$$\delta u = u_\eta + u_{\eta^2}, \quad \delta s = s_\eta + s_{\eta^2}$$

(where the terms in u, s of the order e'^2 are u_{η^2}, s_{η^2}), and separating out those terms which are of the order e'^2. The equation for them becomes, according to the previous notation,

$$\begin{aligned}\zeta^{-1}(D+m)^2 u_{\eta^2} + \zeta^{-1} u_{\eta^2}\Sigma M_i \zeta^{2i} + \zeta s_{\eta^2}\Sigma N_i \zeta^{2i} - \tfrac{3}{8}\zeta^{-2} u_\eta^2 \Sigma P_{-i}\zeta^{2i} \\ - \tfrac{15}{8}\zeta^2 s_\eta^2 \Sigma Q_i \zeta^{2i} - \tfrac{3}{4}\zeta^{-1} u_\eta . \zeta s_\eta \Sigma P_i \zeta^{2i} \\ = \text{coeff. of } e'^2 \text{ in } -\zeta^{-1}[B(u_0+u_\eta)+C(s_0+s_\eta)].\end{aligned}$$

The terms on the right-hand side and the known ones on the left-hand side necessarily contain ζ in the form

$$\Sigma\left[\alpha_i \zeta^{2i} + \beta_i \zeta^{2i+2m} + \beta_i' \zeta^{2i-2m}\right],$$

the solution is therefore of the form

$$u_{\eta^2} = a_0 e'^2 \Sigma_j \left[(\eta\eta')_j \zeta^{2j} + (\eta\eta)_j \zeta^{2j+2m} + (\eta'\eta')_j \zeta^{2j-2m}\right],$$

and a similar expression for s_{η^2}. The coefficients $(\eta\eta')_j$, etc., are the numerical quantities to be found. The method of procedure is the same as before. There are certain multiplications to be performed in the calculation of such a term as $\frac{3}{8}\zeta^{-2}u_\eta^2\Sigma P_{-i}\zeta^{2i}$, but they can be performed in two steps by first computing $\zeta u_\eta . \Sigma P_{-i}\zeta^{2i}$ (this result being wanted in the determination of the terms of order ee') and then multiplying the result by ζu_η.

We can thus proceed to find the terms dependent on all powers of e', m by successive approximation and arrive finally at a solution

$$\begin{aligned}u_{e'} &= u_0 + u_\eta + u_{\eta^2} + \dots\dots, \\ u_{e'} &= s_0 + s_\eta + s_{\eta^2} + \dots\dots\end{aligned}$$

It may be noted that the notation used here is intended to be a suggestive one and to point at once to the order of the terms and to the power of ζ to which they belong. The letter η is associated with e' or the index m of ζ. In the coefficients, η_j is the coefficient of ζ^{2j+m} in u, $(\eta\eta)_j$ or $(\eta^2)_j$ that of ζ^{2j+2m}, etc. This method of notation for the coefficients is found to be a convenient one and it is used for the inequalities all through. For example, if $2q$ be an even positive number,

$$\begin{aligned}u_{\eta^{2q}} = a_0 e'^{2q}\Sigma_j \left[(\eta^q\eta'^q)_j \zeta^{2j} + (\eta^{q+1}\eta'^{q-1})_j \zeta^{2j+2m} + (\eta^{q-1}\eta'^{q+1})_j \zeta^{2j-2m} + \dots\dots \right. \\ \left. + (\eta^{2q})_j \zeta^{2j+2qm} + (\eta'^{2q})_j \zeta^{2j-2qm}\right].\end{aligned}$$

In particular we must notice that the terms in u which do not contain the angle $n't + \epsilon'$ in their argument are

$$a_0 \Sigma_j \left[a_j + e'^2(\eta\eta')_j + e'^4(\eta^2\eta'^2)_j + \dots\dots\right]\zeta^{2j}. \qquad (13)$$

We can suppose now, for theoretical purposes, that all the terms dependent on e', m have been calculated with the necessary accuracy, and by using the numerical value of e' in (13) we can take that for the primary value of u_0 instead of the value (11). But in the actual developments of the lunar theory this simplification has certain disadvantages. It is easier to find the extra small terms due to, for instance, the difference between (13) and (11), than to calculate the corrections to M, N which would be necessary. Hence we commence, when finding the terms depending on the lunar eccentricity or latitude or the parallax of the Sun, by using the primary orbit defined by the equations (11).

5. *The Terms whose Coefficients depend on m and the Eccentricity of the Moon's Orbit. Motion of the Perigee.*

In accordance with the remarks just made, we omit the terms depending on e', and in those results which are of a general character we shall merely state how they may be included and in what way the various expressions are altered by their presence.

We take the equations 9 (ii) and putting

$$u = u_0 + u_e + u_{e^2} + \ldots\ldots,$$
$$s = s_0 + s_e + s_{e^2} + \ldots\ldots,$$

where u_0, s_0 have the values (11), obtain for the determination of u_e, s_e

$$\zeta^{-1}(D+m)^2 u_e + \zeta^{-1} u_e . \Sigma M_i \zeta^{2i} + \zeta s_e . \Sigma N_i \zeta^{2i} = 0.$$

The solution of this equation is known to be of the form

$$\zeta^{-1} u_e = a_0 \Sigma \left[\varepsilon_j \zeta^{2j+c} + \varepsilon'_j \zeta^{2j-c}\right],$$
$$\zeta s_e = a_0 \Sigma \left[\varepsilon'_{-j} \zeta^{2j+c} + \varepsilon_{-j} \zeta^{2j-c}\right].$$

Since c is supposed incommensurable with any whole number, the constant of the argument, an arbitrary of the solution, is not put into evidence. It can be restored at any time if necessary. This remark applies generally.

When the substitution is made in the equation of motion, we obtain a set of linear equations to find ε_j, ε'_j without constant terms, and this fact necessitates a relation between them which determines the quantity c. Since the extension of Adams' theorems proved in Part II, and the results in Part III, depend on this relation to a large extent, it is necessary to develope it somewhat fully and to see in what way c is involved.

Equating to zero the coefficients of $\zeta^{2j\pm c}$, we obtain the series of linear homogeneous equations in ε_j, ε'_j,

$$\left.\begin{array}{l}(2j+1+m+c)^2\,\varepsilon_j \;+\Sigma_i M_i\varepsilon_{j-i} \;+\Sigma_i N_i\varepsilon'_{i-j} = 0,\\ (2j-1-m+c)^2\,\varepsilon'_{-j} + \Sigma_i M_{-i}\varepsilon'_{i-j} + \Sigma_i N_{-i}\varepsilon_{j-i} = 0.\end{array}\right\} \qquad (14)$$

The determinant formed by the elimination of ε_j, ε'_j we may denote by

$$\Delta(c) = \left|\begin{array}{c|c} d(c) & d \\ \hline d & d(-c)\end{array}\right|,$$

where the cross lines denote that d, $d(c)$, $d(-c)$ merely occupy the positions in the determinant assigned above. We have put $d(c)$ for

$$\begin{array}{cccccc}
\cdots & \cdots & \cdots & \cdots & \cdots & \cdots\\
(5+m+c)^2+M_0, & M_{-1}, & M_{-2}, & M_{-3}, & M_{-4}, & \vdots\\
M_1, & (3+m+c)^2+M_0, & M_{-1}, & M_{-2}, & M_{-3}, & \vdots\\
M_2, & M_1, & (1+m+c)^2+M_0, & M_{-1}, & M_{-2}, & \vdots\\
M_3, & M_2, & M_1, & (-1+m+c)^2+M_0, & M_{-1}, & \vdots\\
M_4, & M_3, & M_2, & M_1, & (-3+m+c)^2+M_0, & \vdots\\
\cdots & \cdots & \cdots & \cdots & \cdots & \cdots
\end{array}$$

and $d(-c)$ for the same expression with the sign of c changed. Also d stands for

$$\begin{array}{cccccc}
\cdots & \cdots & \cdots & \cdots & \cdots & \cdots\\
N_0, & N_{-1}, & N_{-2}, & N_{-3}, & N_{-4}, & \vdots\\
N_1, & N_0, & N_{-1}, & N_{-2}, & N_{-3}, & \vdots\\
N_2, & N_1, & N_0, & N_{-1}, & N_{-2}, & \vdots\\
N_3, & N_2, & N_1, & N_0, & N_{-1}, & \vdots\\
N_4, & N_3, & N_2, & N_1, & N_0, & \vdots\\
\cdots & \cdots & \cdots & \cdots & \cdots & \cdots
\end{array}$$

When these are placed in the expression for $\Delta(c)$ in the positions assigned and the cross lines taken away, we have the required determinant.

This is a third form of the infinite determinant giving c. It differs from that obtained by Dr. Hill (II) in being what may be called "doubly-infinite," that is, it is infinite towards the directions not only of the outer sides of the square but also towards those of the cross lines. In this feature it resembles that which I obtained (VIII) by the use of the equations (8), but differs in one most important aspect, namely, that by a suitable system of divisors it can be put into a convergent form.

I repeat the observations there made as to the number of roots and the relations which hold between them. Denoting by q the number of coefficients of the series $\varepsilon_1, \varepsilon_2, \ldots . \varepsilon_\infty$, there are $4q + 2$ unknowns in the $4q + 2$ equations; the determinant has $4q + 2$ rows and columns, and it forms an equation for c of the order $2(4q + 2)$. If c_1, c_2 be two roots of this equation not differing by an even integer, the whole series of roots is

$$\begin{array}{l} \pm c_1, \ \pm(c_1 \pm 2), \ \pm(c_1 \pm 4), \ldots . , \\ \pm c_2, \ \pm(c_2 \pm 2), \ \pm(c_2 \pm 4), \ldots . \end{array}$$

Either c_1 or c_2 is shown to be zero, since

$$u_e = Du, \quad s_e = Ds$$

is a solution of the equations for u_e, s_e.

Now these two sets of roots are also roots of the equation

$$\left(\sin^2 \frac{\pi c}{2} - \sin^2 \frac{\pi c_1}{2}\right)\left(\sin^2 \frac{\pi c}{2} - \sin^2 \frac{\pi c_2}{2}\right) = 0,$$

or of the equation

$$(\cos \pi c - \cos \pi c_1)(\cos \pi c - 1) = 0.$$

Hence we have

$$\Delta(c) \equiv A(\cos \pi c - \cos \pi c_1)(\cos \pi c - 1),$$

an identical equation in which A is independent of c. We most easily find the constant A by equating the coefficients of the highest power of c in each member of the equation. In $\Delta(c)$ this is unity, and in the right-hand side it is

$$\begin{aligned} A \frac{4^2}{1^2} \cdot \frac{4^2}{9^2} \cdot \frac{4^2}{25^2} \ldots . &= A\left(\frac{4}{1} \cdot \frac{4}{3.5} \cdot \frac{4}{7.9} \ldots .\right)^4 \\ &= A\left(\frac{4}{1} \cdot \frac{4}{4^2 - 1} \cdot \frac{4}{8^2 - 1} \ldots .\right)^4, \end{aligned}$$

and therefore

$$\begin{aligned} (\cos \pi c - \cos \pi c_1)(\cos \pi c - 1) &\equiv \Delta(c) \times \left(\frac{4}{1} \cdot \frac{4}{4^2 - 1} \cdot \frac{4}{8^2 - 1} \ldots .\right)^4 \\ &\equiv \nabla(c). \end{aligned}$$

The new determinant $\nabla(c)$ is put into a convenient form by multiplying the middle rows of the upper half of $\Delta(c)$ by 4, the two rows on either side of this by $\frac{4}{4^4 - 1}$, the next two rows above and below by $\frac{4}{8^2 - 1}$ and so on. The lower half being treated in the same way, all the divisors are accounted for. We might then proceed as Dr. Hill has done to obtain the determinant in a better form, with every constituent in the central diagonal unity. The new determinant and

$\nabla(c)$ are both convergent, that is, in each case the product of the constituents of the central diagonal is finite and the sum of the other constituents is finite. (Poincaré IX, Helge von Koch X). But since the determinant will not be used for finding c, and since it has been treated as far as is necessary for the purposes of this paper, it does not seem to be of advantage to develope it further. Two points are of importance in what follows: firstly, *the determinant* $\Delta(c)$ *is a symmetrical one*, and secondly, in consequence of this and also since it vanishes for the special value of c which denotes the principal part of the motion of the lunar Perigee, *the coefficients* ε_j, ε'_j *are proportional to the first minors of the constituents of any row or column.*

It is not difficult to see what changes are made by the introduction of the terms dependent on e'^2 and its powers. Instead of using the values (11) of u_0, s_0, we must use the values (13); also, instead of M_i, N_i we shall have

$$(M)_i = M_i + \text{terms containing } e'^2 \text{ and its powers,}$$
$$(N)_i = N_i + \text{`` \quad `` \quad `` \quad `` \quad ``}$$

No other alteration is necessary in the equations since the terms arising from A, B, C [equation (3)] can be included in (M), (N). We, however, do not have $(M)_i = (M)_{-i}$. All the results concerning the infinite determinant for c still hold. Let for brevity

$$(\varepsilon)_j = \varepsilon_j\,(1 + \text{terms containing } e'^2 \text{ and its powers}),$$
$$(\varepsilon')_j = \varepsilon'_j\,(1 + \text{`` \quad `` \quad `` \quad `` \quad ``}),$$
$$(c) = c + \text{terms containing } e'^2 \text{ and its powers}$$

be the new values. Then as before $(\varepsilon)_j$, $(\varepsilon')_j$ are proportional to the minors of the constituents of any row *or* column of the new determinant, which remains symmetrical.

It remains to state that when c has been found, all the coefficients ε_i, ε'_i can be found in terms of one of them by any suitable method of continued approximation. The minors of the determinant do not seem adapted to calculation. It is found that $\varepsilon_0 - \varepsilon'_0$ is approximately $2e$ where e is Delaunay's eccentricity, and therefore this quantity is a suitable arbitrary. The letter ε is used in all coefficients involving the lunar eccentricity just as the letter η is in all coefficients involving the solar eccentricity.

5. *Further Approximations to the Terms dependent on the Eccentricity of the Moon's orbit and to the Motion of the Perigee.*

The next process is to find the terms of the order e^2. These follow in the same way from the results just obtained as those of order e'^2 did from those of order e'. If the part of c depending on e, before neglected, be δc, we can easily see that δc is of the order e^2 and that we need not know it to obtain the coefficients of the order e^2. We suppose the coefficients of the order e^2 found.

We proceed, then, to find the terms dependent on e^3 with the corresponding new part of c. We are only going to consider the terms depending on the indices $2j \pm (c + \delta c)$ of ζ, since it is by these terms alone that δc is determined. Stopping at the third order we put

$$u = u_0 + u_e + u_{e^2} + u_{e^3},$$

where

$$u_e + u_{e^3} = a_0 \Sigma_j \left[\varepsilon_j \zeta^{2j+c+\delta c} + \varepsilon'_j \zeta^{2j-c-\delta c} + (\varepsilon^2\varepsilon')_j \zeta^{2j+c+\delta c} + (\varepsilon\varepsilon'^2)_j \zeta^{2j-c-\delta c}\right],$$

and similar expressions for the complementary variable. The last two coefficients are of the order e^3, and therefore on substitution in the equation of motion we can put c for $c + \delta c$ when this quantity appears as a factor of these coefficients. We obtain, by separating out the terms of the order e^3,

$$\begin{aligned}&[(2j + 1 + m + c + \delta c)^2 - (2j + 1 + m + c)^2]\, \varepsilon_j \\ &\quad + (2j + 1 + m + c)^2 (\varepsilon^2\varepsilon')_j + \Sigma_i M_i (\varepsilon^2\varepsilon')_{j-i} + \Sigma_i N_i (\varepsilon\varepsilon'^2)_{i-j} \\ &\quad = \text{coeff. of } \zeta^{2j+c+\delta c} \text{ in } \frac{\varkappa\zeta^{-1}}{a_0}(u_0 + u_e + u_{e^2})^{-\frac{1}{2}}(s_0 + s_e + s_{e^2})^{-\frac{3}{2}} \text{ of the order } e^3,\end{aligned}$$

and a similar equation. Omitting negligible quantities on the left-hand side, the equation becomes

$$\begin{aligned}&2\delta c\,(2j + 1 + m + c)\,\varepsilon_j + (2j + 1 + m + c)^2 (\varepsilon^2\varepsilon')_j + \Sigma_i M_i (\varepsilon^2\varepsilon')_{j-i} + \Sigma_i N_i (\varepsilon\varepsilon'^2)_{i-j} \\ &\qquad = \text{same expression as before.} \qquad (15)\end{aligned}$$

Similarly the other equation gives

$$\begin{aligned}&2\delta c\,(2j - 1 - m + c)\,\varepsilon'_{-j} + (2j - 1 - m + c)(\varepsilon\varepsilon'^2)_{-j} + \Sigma_i M_{-i} (\varepsilon\varepsilon'^2)_{i-j} + \Sigma_i N_{-i} (\varepsilon^2\varepsilon')_{j-i} \\ &\quad = \text{coeff. of } \zeta^{-2j-c-\delta c} \text{ in } \frac{\varkappa\zeta^{-1}}{a_0}(u_0 + u_e + u_{e^2})^{-\frac{1}{2}}(s_0 + s_e + s_{e^2})^{-\frac{3}{2}} \text{ of the order } e^3. \quad (15)\end{aligned}$$

The terms on the right-hand sides of these equations are known, those on the left-hand sides are to be determined. We can get an equation of the first order for δc free from the as yet undetermined coefficients $(\varepsilon^2\varepsilon')$, $(\varepsilon\varepsilon'^2)$. There are, in the notation of the last section, $4q + 2$ equations and $4q + 3$ unknowns to be determined. One of these is arbitrary, and it involves a new definition of the "eccentricity constant" of integration. We may define this as may be most convenient. At present we proceed to the determination of δc, which is not affected by this observation.

Multiply the first of equations (15) by ε_j, the second by ε'_{-j}, and sum the two for all values of j. Now, in these equations the coefficients of $(\varepsilon^2\varepsilon')_j$, $(\varepsilon\varepsilon'^2)_{-j}$ are exactly the same, term for term, as those of ε_j, ε'_{-j} in equations (14). If, therefore, we multiply the equations (15) by the respective minors of $\Delta(c)$, all terms involving $(\varepsilon^2\varepsilon')_j$, $(\varepsilon\varepsilon'^2)_{-j}$ will disappear. But as these minors are proportional to ε_j, ε'_{-j}, the same object will be attained if we multiply by ε_j, ε'_{-j}. If, however, we conceive of difficulties arising from the fact that $\Delta(c)$ is not in a convergent form and therefore that its minors are infinite quantities, though their ratios are finite, we can proceed as follows. After the multiplication by ε_j, ε'_{-j} and the addition of the equations for all values of j, the coefficient of $(\varepsilon^2\varepsilon')_i$ is

$$(2i + 1 + m + c)^2\,\varepsilon_i + \Sigma_j M_j \varepsilon_{i-j} + \Sigma_j N_j \varepsilon'_{j-i},$$

which, by equations (14), is zero. And so for all the other similar terms. This process is, of course, algebraically identical with the previous one.

Hence we have

$$\begin{aligned}2\delta c\,\Sigma_j\,&[(2j + 1 + m + c)\,\varepsilon_j^2 + (2j - 1 - m + c)\,\varepsilon'^2_{-j}]\\ &= \Sigma_j\Big[\varepsilon_j \times \text{coeff. of } \zeta^{2j+c+\delta c} \text{ in } \frac{\varkappa\zeta^{-1}}{a_0}(u_0 + u_e + u_{e^2})^{-\frac{1}{2}}(s_0 + s_e + s_{e^2})^{-\frac{3}{2}}\\ &\quad + \varepsilon'_{-j} \times \text{coeff. of } \zeta^{-2j-c-\delta c} \text{ in } \frac{\varkappa\zeta^{-1}}{a_0}(u_0 + u_e + u_{e^2})^{-\frac{1}{2}}(s_0 + s_e + s_{e^2})\Big]^{-\frac{3}{2}},\end{aligned}$$

where the terms on the right-hand side of the order e^4 alone are considered. But

$$s_e\zeta = a_0\Sigma_j\,[\varepsilon_j\zeta^{-2j-c-\delta c} + \varepsilon'_{-j}\zeta^{2j+c+\delta c}],$$

and therefore, using this on the right-hand side of the previous equation, we obtain

$$\begin{aligned}2\delta c\,\Sigma_j\,&[(2j + 1 + m + c)\,\varepsilon_j^2 + (2j - 1 - m + c)\,\varepsilon'^2_{-j}]\\ &= \text{const. part, order } e^4, \text{ in } \frac{\varkappa}{a_0^2}\cdot\frac{(u_0 + u_e + u_{e^2})\,s_e}{(u_0 + u_e + u_{e^2})^{\frac{3}{2}}(s_0 + s_e + s_{e^2})^{\frac{3}{2}}}. \quad (16)\end{aligned}$$

Transforming to real coordinates and putting

$$X_{e^2} = x_0 + x_e + x_{e^2}, \quad Y_{e^2} = y_0 + y_e + y_{e^2}, \quad R^2_{e^2} = X^2_{e^2} + Y^2_{e^2},$$

the right-hand side becomes

$$\text{const. part, order } e^4, \text{ in expansion of } \frac{\varkappa}{a_0^2} \cdot \frac{X_{e^2}x_e + Y_{e^2}y_e}{R^3_{e^2}}, \tag{16$'$}$$

an expression of somewhat remarkable form and simplicity. When first obtained, it seemed to suggest that Adams' theorems (IV) as to the connection between the constant parts of the Parallax of the Moon and certain parts of the motions of the Perigee and Node, must really arise naturally from this mode of development of the lunar theory. After some trouble I succeeded in proving again these theorems and extending them so as to show them not merely in the form of ratios but in an exact form. They will be given in Part II of this paper.

It is easy to extend these results so as to include all powers of e'. When we do so the formula (16) or (16′) becomes

$$2\delta(c)\Sigma_j\left[\{2j+1+m+(c)\}(\varepsilon)^2_j + (2j-1-m+c)(\varepsilon')^2_{-j}\right]$$
$$= \text{const. part, order } e^4, \text{ in } \frac{\varkappa}{a_0^2} \cdot \frac{(X)_{e^2}(x)_e + (Y)_{e^2}(x)_e}{(R)^3_{e^2}}, \tag{17}$$

where the brackets round the various symbols have the same meaning as that given at the end of section 4.

The coefficient of δc or $\delta(c)$ remains the same if we are finding the terms in the motion of Perigee dependent on the latitude of the Moon or the parallax of the Sun. If we are in the process of finding $\delta(c)$, that part that depends on e'^2 can be obtained as in equation (16) by finding the constant of order $e^2e'^2$ in the expansion of

$$\frac{\varkappa}{a_0^2} \cdot \frac{(X)_e x_e + (Y)_e y_e}{(R)^3_{e^2}},$$

where $(X)_e$, $(Y)_e$, $(R)_e$ contain merely the terms, supposed to be previously found, dependent on e, e', e'^2, ee' and m. There are also the terms arising from ω_2 to be included. So that in all cases the determination of the new part of c depends only on an equation which contains known terms found in previous approximations.

Similar results will follow for the terms dependent on higher powers of e^2, e'^2 in c. For example, the equation necessary to find the part in c of the order e^4 may be easily obtained by looking at equations (15). It is $\delta^2 c$ where $\delta^2 c$ is given by

$$\left.\begin{aligned} 2\delta^2 c\,\Sigma_j\left[(2j+1+m+c)\,\varepsilon_j^2+(2j-1-m+c)\,\varepsilon'^2_{-j}\right]+(\delta c)^2\,\Sigma_j\left[\varepsilon_j^2+\varepsilon'^2_{-j}\right] \\ +\,\Sigma_j\left[(2j+1+m+c)^2\,\varepsilon_j(\varepsilon^2\varepsilon')_j+(2j-1-m+c)^2\,\varepsilon'_{-j}\,(\varepsilon\varepsilon'^2)_{-j}\right] \\ =\text{const. coeff., order } e^4,\text{ in }\frac{\varkappa}{a_0^2}\cdot\frac{X_{e^4}x_e+Y_{e^4}y_e}{R_{e^4}^3}, \end{aligned}\right\}\quad(18)$$

where $$X_{e^4}=x_0+x_e+x_{e^2}+x_{e^3}+x_{e^4},\text{ etc.}$$

The coefficients to be determined therewith by the equations from which this expression arises, are those of the order e^5 with arguments $(2iD \pm l)$, where $2D$ is the argument of the Variation and l that of the principal elliptic term.

If we wish to determine the terms dependent on the square of the parallax of the Sun, similar results will follow. On the right-hand sides of the equations we shall have further terms arising from Ω_1, but these terms will be all known by previous computations, so that the general result for all terms in the motion of the perigee is this: *In any stage of the approximations we can always find the new part of the motion of the perigee by simple computations without being compelled either to solve an infinite determinant or to find by successive approximation the new coefficients arising therewith. The new coefficients can then be found by the usual methods.*

These results hold for any terms whatever in the motion of the perigee, and also, as will be seen immediately, for those in the motion of the node. The most important feature in them all is that just stated in italics. The solution of an infinite determinant, in whatever way performed, is very laborious, and the continued approximation method applied to δc and to the coefficients simultaneously, is not much less so. The method here given, when once the parts dependent on m only have been calculated, avoid these laborious processes—an important point quite independent of its mathematical interest. It is not to be forgotten also that in one respect theory here goes hand in hand with observation. The motions of perigee and node are now capable of being determined by observation with much greater accuracy than the majority of the coefficients of the periodic terms. In the method outlined above theory also gives the new parts of the motions before the coefficients of the corresponding periodic terms.

We now proceed to find the effect of including in our equations the motion of the Moon in latitude.

6. *Terms dependent on the Latitude of the Moon. Motion of the Node.*

For these terms we use equations 9 (iii), omitting those dependent on the solar eccentricity. Since z is a small quantity in powers of which we suppose that expressions are possible, and since the second of these equations contains it in its first, third, fifth, ... powers only, while the first equation contains it in its second, fourth, ... powers only, we are able to treat the equations separately, going first to one and then to the other as we obtain each new approximation. This fact facilitates the performance of the actual calculations very largely.

Let γ denote generally a quantity of the same order as z. The z-equation is

$$D^2 z - \left(m^2 + \frac{\varkappa}{(r^2+z^2)^{\frac{3}{2}}}\right) z = 0.$$

Let

$$\left.\begin{aligned} z &= z_\gamma + z_{\gamma^3} + z_{\gamma^5} + \ldots., \\ u &= u_0 + u_{\gamma^2} + u_{\gamma^4} + \ldots., \\ s &= s_0 + s_{\gamma^2} + s_{\gamma^4} + \ldots.. \end{aligned}\right\} \tag{19}$$

Expanding the last term in the z-equation, we have

$$D^2 z - \left(m^2 + \frac{\varkappa}{r^3}\right) z = -\frac{3}{2}\frac{\varkappa z^3}{r^5} + \frac{15}{8}\frac{\varkappa z^5}{r^7} + \ldots.$$

The equation for z_γ will be

$$D^2 z_\gamma - \left(m^2 + \frac{\varkappa}{r_0^3}\right) z_\gamma = 0. \tag{20}$$

The solution of this equation can be put into the form

$$z_\gamma \sqrt{-1} = a_0 \Sigma \left[K_j \zeta^{2j+g} + K'_{-j}\zeta^{-2j-g}\right], \qquad j = +\infty \ldots. -\infty, \tag{21}$$

where g must be found, and K_j, K'_{-j} are of the order γ and such that $K_j = -K'_{-j}$. The arbitrary constant attached to the new angle which the presence of g introduces can be understood, as in section 4, to be present.

It is known from Adams' work (III) that g is found from an infinite determinant. Expanding $\varkappa / r_0^3$, we have from section 2

$$D^2 z_\gamma - 2 z_\gamma \Sigma M_i \zeta^{2i} = 0,$$

and therefore on substitution of the assumed value of z_γ and on equating the coefficients of ζ^{2i+g} to zero,

$$(2j+g)^2 K_j - 2\Sigma K_i M_{j-i} = 0. \tag{22}$$

This set of equations gives a symmetrical infinite determinant which, after certain transformations, was calculated by Adams (III) to find g. The coefficients K_j are proportional to the first minors of the constituents of any row *or* column of this determinant. When g is known and we have thence obtained the ratios of K_j to any one of them, say K_0, which we take as the arbitrary constant of the solution, we substitute the resulting values of z in the first of equations 9 (iii) and find by the same methods as before the values of u_{γ^2}, s_{γ^2}, which, as the suffixes indicate, are of the order γ^2.

Returning with these to the z-equation, it is desired to find the terms in z of the order γ^3 together with those in g of the order γ^2. Confining our attention to the latter object, we assume

$$(z_\gamma + z_{\gamma^3})\sqrt{-1} = a_0 \Sigma \left[\{K_j + (K^2K')_j\}\, \zeta^{2j+g+\delta g} + \{K'_{-j} + (KK'^2)_{-j}\}\, \zeta^{-2j-g-\delta g}\right].$$

We omit the terms which depend on the indices $2i \pm 3g$, as their coefficients can be determined to the order γ^3 with the known value of g. We have $(K^2K')_j = -(KK'^2)_{-j}$. Equating to zero the coefficients of $\zeta^{2j+g+\delta g}$ for all values of j, we get a set of equations for the determination of the new coefficients and δg. Let

$$X_{\gamma^2} = x_0 + x_{\gamma^2}, \quad Y_{\gamma^2} = y_0 + y_{\gamma^2}, \quad R^2_{\gamma^2} = X^2_{\gamma^2} + Y^2_{\gamma^2}, \tag{23}$$

then, by reasoning similar to that used in the determination of δc, the set of equations may be written

$$\begin{aligned} 2\delta g\,(2j+g)\,K_j + (2j+g)^2\,(K^2K')_j - 2\Sigma M_{j-i}\,(K^2K')_i & \\ = \text{coeff., order } \gamma^3, \text{ of } \zeta^{2j+g+\delta g} \text{ in } \frac{\varkappa z_\gamma\sqrt{-1}}{a_0\,(R^2_{\gamma^2}+z^2_\gamma)^{\frac{3}{2}}}. & \qquad (24)\end{aligned}$$

Multiplying this equation by K_j, and summing the set of equations so treated for all values of j, we have by equation (22) (as in the last section) the new coefficients disappearing and

$$2\delta g \Sigma\,(2j+g)\,K_j^2 = \left[\Sigma\, K_j \times \text{coeff., order } \gamma^4, \text{ of } \zeta^{2j+g+\delta g} \text{ in } \frac{\varkappa z_\gamma\sqrt{-1}}{a_0\,(R^2_{\gamma^2}+z^2_\gamma)^{\frac{3}{2}}}\right]. \tag{25}$$

Similarly,

$$2\delta g \Sigma\,(2j+g)\,K'^2_{-j} = \Sigma \left[K'_{-j} \times \text{coeff., order } \gamma^4, \text{ of } \zeta^{-2j-g-\delta g} \text{ in } \frac{\varkappa z_\gamma\sqrt{-1}}{a_0\,(R^2_{\gamma^2}+z^2_\gamma)^{\frac{3}{2}}}\right]. \tag{25}$$

Now, since $K'_{-j} = -K_j$, and therefore

$$-z_\gamma\sqrt{-1} = a_0 \Sigma \left[K_j \zeta^{-2j-g-\delta g} + K'_{-j} \zeta^{2j+g+\delta g}\right],$$

we obtain, by addition of the two previous equations,

$$4\delta g\,\Sigma\,(2j+g)\,K_j^2 = \text{const. coeff., order } \gamma^4, \text{ in } \frac{\varkappa z_\gamma^2}{a_0^2\,(R_{\gamma^2}^2+z_\gamma^2)^{\frac{3}{2}}}, \tag{26}$$

which is the formula for obtaining δg.

This formula is easily extended so as to include all powers of e'. In determining K_j, instead of using r_0 we use (r_0), where the notation is that used in section 5. Let the corresponding values of K_j, g, δg, etc., be $(K)_j$, (g), $\delta(g)$, etc. The equation then becomes

$$4\delta(g)\,\Sigma\,\{2j+(g)\}\,(K)_j^2 = \text{const. coeff., order } \gamma^4, \text{ in } \frac{\varkappa\,(z)_\gamma^2}{a_0^2\,\{(R)_{\gamma^2}^2+(z)_\gamma^2\}^{\frac{3}{2}}}. \tag{27}$$

The part of the motion of the node which is of the order e^2, and that of the motion of the perigee which is of the order γ^2, will be required for the investigations in Part II. They are not difficult to write down if we look at the results (16′) and (26). Let $z_{e\gamma}$ be the part of z of the order $e\gamma$. This can be found when we know the principal parts of the motions of perigee and node only. It can then be easily seen that the part of δg depending on e^2 is given by means of the equation

$$4\delta g\,\Sigma_j\,(2j+g)\,K_j^2 = \text{const coeff., order } e^2\gamma^2, \text{ in } \frac{\varkappa\,z_\gamma\,(z_\gamma+z_{e\gamma})}{a_0^2\,R_{e^2}^2}. \tag{28}$$

Also that the part of δc of the order γ^2 is given by the equation

$$\begin{aligned}2\delta c\,\Sigma_j\,&[(2j+1+m+c)\,\varepsilon_j^2+(2j-1-m+c)\,\varepsilon_{-j}'^2]\\ &= \text{const. part., order } e^2\gamma^2, \text{ in } \frac{\varkappa}{a_0^2}\cdot\frac{(X_e+x_{\gamma^2})\,x_e+(Y_e+y_{\gamma^2})\,y_e}{[(X_e+x_{\gamma^2})^2+(Y_e+y_{\gamma^2})^2+(z_\gamma+z_{e\gamma})^2]^{\frac{3}{2}}}.\end{aligned} \tag{29}$$

Finally, by putting brackets ( ) round each letter, we may include as before in the formulæ (28) and (29), all terms dependent on e'^2 and its powers.

The equations (28) and (29), with powers of e'^2 included, will be named (30) and (31) respectively. (30), (31)

It is unnecessary to go further with these expansions. Sufficient has been said to indicate the manner of treating the whole Lunar Theory after this method. The terms which depend on the parallax of the Sun are a little troublesome owing to the new terms which arise from Ω_1, but they are much less so than when treated by the simultaneous equations (8), since, for a given degree of approximation, they are one order lower in u, s in the former case than in the latter. Nevertheless, should it be considered that in finding the terms of high orders in e, e', γ, $1/a'$ the method indicated above becomes too troublesome, nothing prevents us from returning to equations (8), these being available at any stage. No increase of labor results from the mere change of method.

After the experience gained in the calculations of the coefficients made up to the present time by both methods, it seems probable that the equations (8) will involve less labor than (9) for terms of the fourth order in e, e', γ^2, $1/a'$, if not for the third. Nothing even then stands in the way of the determination of the new parts of c and g by the methods just exhibited.

7. *Order of Proceeding.*

Having resolved the chief difficulties in the theory, namely, the determination of the motions of Perigee and Node, it is possible to give a consistent plan for the computation of the complete expressions of the coordinates. The order which seems to arise most naturally from the previous results may be sketched out as follows:

Terms dependent on

(i) m only.
(ii) m and e', e'^2, e'^3 .... in succession.
(iii) m and e, ee', ee'^2, ..., e^2, e^2e', $e^2e'^2$, ..., e^3, e^3e', ...
(iv) m and γ, $\gamma e'$, $\gamma e'^2$, ..., γ^2, $\gamma^2 e'$, ..., γ^3, ...
(v) m and any other combinations of γ, e, e'.
(vi) m and $1/a'$ and combinations with γ, e, e'.

This order may be varied somewhat.

The arguments in Delaunay's notation are, $2D$ that of the variation, l that of the principal elliptic inequality, l' that of the annual equation, F that of the principal term in latitude. The expressions of the coordinates should then be arranged in the order

$$2iD,\ 2iD \pm l',\ 2iD \pm 2l', \ldots,\quad 2iD \pm l,\ 2iD \pm l \pm l',\ 2iD \pm l \pm 2l', \ldots,$$

and so on. The chief difference from Delaunay's arrangement being that the terms whose arguments differ only by multiples of $2D$ are to be placed together in all cases.

It is proposed to carry the computations sufficiently far in x, y, z to enable the coefficients in longitude and latitude to be found accurately to one thousandth of a second of arc. This makes it necessary to carry some of the earlier calculations to a somewhat greater degree of accuracy than is actually needed for the special coefficients. But in any case the terms in m only, and the principal parts of the motions of perigee and node, carried to 15 places of decimals by Hill and Adams (I, II, III), are amply sufficient.

8. *The Arbitrary Constants.*

The constants which arise in the arguments by the integrations are the same as those of Delaunay. Those outside of the signs, sine and cosine have been referred to generally as a_0, e, γ. As for a_0, we can define it in such a way that the determination of the coefficients is made as simple as possible, and this will cause in general a determination of a new part to a_0 whenever we are making an approximation which contains a constant term. It would seem better, however, to define a_0 as that value given by Dr. Hill (I) in terms of $(\mu/n^2)^{\frac{1}{3}}$ and m, making the necessary changes after each approximation when the latter has been made under the previous method. Similarly, it seems the best to define the "eccentricity" constant so that it is the coefficient of $\sin l$ in x/a_0, and the latitude constant so that it is the coefficient of $\sin F$ in z/a_0. Then a_0 differs from Delaunay's a by terms which vanish with m; the first approximation (neglecting m and all powers and products of the constants) to the eccentricity constant is $2e$, where e is Delaunay's constant; and the similar first approximation to the latitude constant is the quantity defined by Delaunay as γ.

The numerical value of m is used all through, the other constants being left arbitrary.

9. *Verification of the Results.*

For verification purposes we have several equations at our disposal. The most useful of these when the coefficients of ζ^{2j} are not under consideration, is the second of equations (8). If it be desired to verify the coefficients of $\zeta^{2j+\Lambda}$, where Λ is some linear combination of m, c, g, we substitute in this equation the results obtained and find therefrom an equation of the form

$$\lambda\zeta^{\Lambda} + \lambda'\zeta^{-\Lambda} = 0,$$

where λ, λ' contain integral powers of ζ only. The coefficients of the various powers of ζ ought to vanish identically after substitution, and therefore if we put $\zeta = 1$ in λ, λ', the results ought equally to be zero identically. When $\zeta = 1$, $\lambda + \lambda' = 0$, and therefore this method will not be available when A is zero. In this equation of verification all the terms are of the second degree in u, s, and the combinations of the coefficients are totally different from those arising from equations (4) used for finding them.

As for the coefficients of ζ^{2j}, the results proved in Part II concerning the constant part of $1/r$ used in connection with equation (6) will verify all coefficients of the orders e^2, γ^2, e^4, $e^2\gamma^2$, γ^4 and all powers e'^2. For practical purposes, then, we have only the terms of orders e'^2, e'^4, ... $1/a'^2$, e^2/a'^2, ..., etc., outstanding. Equation (6) can be used for these by expanding $2\varkappa(us + z^2)$, and, after choosing out the terms of the required order, putting $\zeta = 1$.

10. *General Observations.*

It will be well to contrast the method here outlined with that used in the development of the Elliptic and Parallactic Inequalities (VII, VIII). It must be premised that both methods are built up on Dr. Hill's primary solution. These remarks refer merely to the best method of continuation.

The chief disadvantage of the present treatment lies in the frequent multiplication of series of ζ with numerical coefficients. Although every multiplication is available for several classes of inequalities, besides the one for which it was especially made, in terms of high order these computations will become very tedious. But there are several advantages which may render the method worthy of consideration. In the first place there are no numerical coefficients to be computed after the multiplication of series has been performed, like those arising from the use of equations (8). In the latter, for instance, when finding the terms of order e^2 we have to compute

$$\Sigma_j (j, i, 0, 1)\, \epsilon_j \epsilon'_{j-i}$$

for all values of i, the portion in brackets being a somewhat complicated function of m and c, and different for different values of i, j (VIII, p. 322). Such calculations are fruitful in producing errors. Secondly, the terms arising from Ω_1 are very simple and, when $1/a'$ is neglected, of the first order in u, s, so that only a very few short operations are necessary. Thirdly—perhaps the most important point to be mentioned—the multiplications of series, being always of the same nature, can be performed by a computer from one general formula. This would require little extra knowledge on his part beyond that of logarithms, and his results can be checked by a computation with a special value of ζ.

The method here outlined has been successfully employed in computing the coefficients of orders e', e'^2, ee' with the numerical value of m;* much labor apparently was saved by the readiness with which the work was reduced to easily remembered operations and by the diminished opportunities of making errors in calculation. Certain difficulties owing to the near equality of the coefficients of $\zeta^{\pm m}$ and $\zeta^{\pm(2-m)}$ and also to the large divisors, were more readily bridged. I hope shortly to test the terms in e^2, and also the part of the motion of the perigee of the order e^2 (VIII, p. 337). This last differed somewhat from Delaunay's value (VI), and before it can be accepted, ought to have a further test applied and the errors, if any, corrected.

PART II.

THEOREMS CONCERNING THE CONSTANT PART IN THE EXPRESSION OF THE PARALLAX OF THE MOON.

In the investigations which follow, the Parallax of the Sun is neglected. With this exception the results obtained are perfectly general.

Let
$$x = X_q, \quad y = Y_q, \quad z = Z_q$$
be the general solution of equations (9) or (10), neglecting quantities in the coefficients beyond the order $e^p\gamma^{q-p}$ $(p = 0, 1, 2 \ldots . q)$. We define e, γ as the arbitrary constants of integration in eccentricity and latitude, of the same order of small quantities as those used by Delaunay.

Let
$$\left.\begin{aligned} X_q &= x_0 + x_1 + \ldots . + x_q, \\ Y_q &= y_0 + y_1 + \ldots . + y_q, \\ Z_q &= z_1 + z_2 + \ldots . + z_q, \\ R_q^2 &= X_q^2 + Y_q^2 + Z_{q-1}^2 , \end{aligned}\right\} \qquad (32)$$
where x_q, y_q are of the form $\alpha_q e^q + \alpha_{q-2} e^{q-2}\gamma^2 + \alpha_{q-4} e^{q-4}\gamma^4 + \ldots .$, and z_q is of the form $\alpha_{q-1} e^{q-1}\gamma + \alpha_{q-3} e^{q-3}\gamma^3 + \ldots .$

From what we know of the expressions of the coordinates of the Moon, α_q, α_{q-2}, ... are functions of m, e' and of the angles $2\nu(t - t_0)$, $n't + \varepsilon'$,

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* Monthly Not. R. A. S., Vols. LIV, p. 471; LV, pp. 3-5.

$cv(t-t_1)$, $2gv(t-t_2)$ and $\alpha_{q-1}, \alpha_{q-3} \ldots.$ of m, e' and the angles $2v(t-t_0)$, $n't+\varepsilon'$, $cv(t-t_1)$ and odd multiples of $gv(t-t_2)$. According to the notation previously used there should be brackets ( ) round all the symbols X, x, etc. These must be understood to be present, but are omitted here for the sake of simplicity of expression. We proceed to prove some theorems.

11. *If* x, y, z *have been calculated to the order* $2q-2$, *the constant part in the expansion of* $1/\sqrt{(x^2+y^2+z^2)}$ *can be obtained to the order* $2q$ *without further reference to the equations of motion.*

With the notation just outlined, this theorem may be more fully stated by means of the equation,

$$\left.\begin{array}{l} \text{const. part, order } 2q\text{, in the expansion of } \dfrac{3}{R_{2q}} \\ \qquad = \text{const. part, order } 2q\text{, in the expansion of} \\ \qquad\qquad \left[\dfrac{3}{R_{2q-2}} - \dfrac{x_0 X_{2q-2} + y_0 Y_{2q-2}}{R^3_{2q-2}} - 3\,\dfrac{z_1 z_{2q-2}}{r_0^5}(x_0x_1 + y_0y_1)\right]. \end{array}\right\} \quad (33)$$

The proof of this is based on Adams' method of proving his theorems referred to below (IV). The equations (10) are, if $r^2 = x^2 + y^2$,

$$\left.\begin{array}{l} \ddot{x} - 2n'\dot{y} - 3n'^2x + A'x + B'y = -\dfrac{\mu x}{(r^2+z^2)^{\frac{3}{2}}}, \\ \ddot{y} + 2n'\dot{x} \qquad\qquad + B'x + C'y = -\dfrac{\mu y}{(r^2+z^2)^{\frac{3}{2}}}, \\ \ddot{z} \qquad + n'^2z + n'\rho z \qquad\qquad = -\dfrac{\mu z}{(r^2+z^2)^{\frac{3}{2}}}. \end{array}\right\} \quad (34)$$

Let x_0, y_0, z_0 be the values of x, y, z, expressed by the same formulæ and with the same angles, but with different arbitrary constants of eccentricity and inclination (i. e. different values of e, γ). We have then

$$\left.\begin{array}{l} x_0 - 2n'\dot{y}_0 - 3n'^2x_0 + A'x_0 + B'y_0 = -\dfrac{\mu x_0}{(r_0^2+z_0^2)^{\frac{3}{2}}}, \\ \ddot{y}_0 + 2n'\dot{x}_0 \qquad\qquad + B'x_0 + C'y_0 = -\dfrac{\mu y_0}{(r_0^2+z_0^2)^{\frac{3}{2}}}, \\ \qquad + n'^2z_0 + n'^2\rho z_0 \qquad\qquad = -\dfrac{\mu z_0}{(r_0^2+z_0^2)^{\frac{3}{2}}}. \end{array}\right\} \quad (35)$$

Let the suffix 0 correspond to zero values of e, γ. We suppose as before that expansion may be made in powers of e, γ. Since equations (34) hold generally, they must hold if, after the right-hand sides have been expanded, we stop at any given order in e or γ in all terms. They will therefore hold if we stop at the order $2q$ in e, γ. Hence we can put $x = X_{2q}$, $y = Y_{2q}$, $z = Z_{2q-1}$, as Z^2_{2q-1} is the value of z correct to the order $2q$. Multiply the first two of equations (34) by x_0, y_0 respectively and subtract the sum of the results from the sum of the first two of equations (35) multiplied by X_{2q}, Y_{2q} respectively. We can put the resulting equation into the form

$$\begin{aligned}\frac{d}{dt}\left[\dot{X}_{2q}x_0 + \dot{Y}_{2q}y_0 - \dot{x}_0X_{2q} - \dot{y}_0Y_{2q} - 2n'(Y_{2q}x_0 - X_{2q}y_0)\right]&\\ = -\mu(X_{2q}x_0 + Y_{2q}y_0)\left(\frac{1}{R^3_{2q}} - \frac{1}{R^3_0}\right)&.\end{aligned}$$

Since the expressions of the coordinates contain t only under the signs, sine cosine, we deduce immediately,
const. part to the order $2q$ inclusive, in the expansion of

$$(X_{2q}x_0 + Y_{2q}y_0)\left(\frac{1}{R^3_{2q}} - \frac{1}{R^3}\right) = 0. \qquad (36)$$

Let the notation $C_{2q}[Q]$ denote "the constant part, order $2q$, in the expansion of Q" according to any parameter—which we can put into evidence when it is necessary for the sake of clearness. The equation (36) then gives

$$\begin{aligned}C_{2q}\left[\frac{X_{2q}x_0 + Y_{2q}y_0}{R^3_{2q}}\right] &= C_{2q}\left[\frac{X_{2q}x_0 + Y_{2q}y_0}{R^3_0}\right]\\ &= C_{2q}\left[\frac{x_{2q}x_0 + y_{2q}y_0}{r^3_0}\right], \qquad (37)\end{aligned}$$

since the parts of X_{2q}, Y_{2q} of order $2q$ are x_{2q}, y_{2q} and $R^2_0 = X^2_0 + Y^2_0 = x^2_0 + y^2_0 = r^2_0$. The equation (37) holds for any positive integral value of q.

Now, if we stop at the order $2q$,

$$R^2_{2q} = R^2_{2q-1} + 2x_0x_{2q} + 2y_0y_{2q} + 2z_1z_{2q-1},$$

and therefore

$$\begin{aligned}C_{2q}\left[\frac{x_0X_{2q} + y_0Y_{2q}}{R^3_{2q}}\right] = C_{2q}\Bigg[&\frac{x_0X_{2q-1} + y_0Y_{2q-1}}{R^3_{2q-1}} + \frac{x_0x_{2q} + y_0y_{2q}}{r^3_0}\\ &- 3(x^2_0 + y^2_0)\frac{x_0x_{2q} + y_0y_{2q} + z_1z_{2q-1}}{r^5_0}\Bigg]. \qquad (38)\end{aligned}$$

Putting r_0^2 for $x_0^2 + y_0^2$, we deduce from (37) and (38)

$$C_{2q}\Big[\frac{x_0 X_{2q-1} + y_0 Y_{2q-1}}{R_{2q-1}^3}\Big] = C_{2q}\Big[3\,\frac{x_0 x_{2q} + y_0 y_{2q} + z_1 z_{2q-1}}{r_0^3}\Big]. \tag{39}$$

Again, by expansion,

$$C_{2q}\Big[\frac{1}{R_{2q}}\Big] = C_{2q}\Big[\frac{1}{R_{2q-1}} - \frac{x_0 x_{2q} + y_0 y_{2q} + z_1 z_{2q-1}}{r_0^3}\Big].$$

Hence by (39),

$$\begin{aligned} C\ \Big[\frac{3}{R_{2q}}\Big] &= C_{2q}\Big[\frac{3}{R_{2q-1}} - \frac{x_0 X_{2q-1} + y_0 Y_{2q-1}}{R_{2q-1}^3}\Big] \\ &= C_{2q}\Big[\frac{3}{(R_{2q-2}^2 + 2S_{0,\,2q-1} + 2S_{1,\,2q-1})^{\frac{1}{2}}} \\ &\qquad - \frac{x_0 X_{2q-2} + y_0 Y_{2q-2} + S_{0,\,2q-1} - z_1 z_{2q-2}}{(R_{2q-2}^2 + 2S_{0,\,2q-1} + 2S_{1,\,2q-1})^{\frac{3}{2}}}\Big], \end{aligned}$$

where we have put

$$\begin{aligned} S_{0,\,2q-1} &= x_0 x_{2q-1} + y_0 y_{2q-1} + z_1 z_{2q-2}, \\ S_{1,\,2q-1} &= x_1 x_{2q-1} + y_1 y_{2q-1} + z_1 z_{2q-2}. \end{aligned}$$

Hence by expansion we obtain

$$\left.\begin{aligned} C_{2q}\Big[\frac{3}{R_{2q}}\Big] &= C_{2q}\Big[\frac{3}{R_{2q-2}} + 9\,\frac{S_{0,\,1} S_{0,\,2q-1}}{r_0^5} - 3\,\frac{S_{1,\,2q-1}}{r_0^3} - \frac{x_0 X_{2q-2} + y_0 Y_{2q-2}}{R_{2q-2}^3} \\ &\qquad + 3S_{0,\,1}\,\frac{S_{0,\,2q-1} - z_1 z_{2q-2}}{r_0^5} \\ &\qquad - (r_0^2 + S_{0,\,1})\Big(-3\,\frac{S_{0,\,2q-1}}{r_0^5} + 15\,\frac{S_{0,\,2q-1} S_{0,\,1}}{r_0^7} - 3\,\frac{S_{1,\,2q-1}}{r_0^5}\Big)\Big] \\ &= C_{2q}\Big[\frac{3}{R_{2q-2}} - \frac{x_0 X_{2q-2} + y_0 Y_{2q-2}}{R_{2q-2}^3} - 3\,\frac{z_1 z_{2q-2} S_{0,\,1}}{r_0^5}\Big)\Big], \end{aligned}\right\} \tag{33'}$$

which formula, since $z_0 = 0$, proves the theorem.

12. *Adams' First Theorem.*

We can deduce this theorem immediately from the above result. It states *that in the constant part of the Moon's Parallax all the terms of orders e^2 and γ^2 are zero.*

In equation (33′) let $q = 1$. We obtain, since $z_0 = 0$,

$$C_2\Big[\frac{3}{R_2}\Big] = C_2\Big[\frac{3}{R_0} - \frac{x_0^2 + y_0^2}{r_0^3}\Big] = C_2\Big[\frac{2}{r_0}\Big] = 0.$$

First, let $\gamma = 0$; this shows that the term in e^2 is zero. Next, let $e = 0$, then similarly the term in $\gamma^2 = 0$. Also it is to be remembered that e, γ occur only in the const. part of $1/R$ in even powers. In the above, e'^2 and its powers are included.

13. *Adams' Second Theorem and its Extension.*

This theorem may be stated as follows:

If

$$C_4\left[\frac{1}{R}\right] = Ee^4 + 2Fe^2\gamma^2 + G\gamma^4 \tag{40}$$

and

$$C_2\,[(c)] = He^2 + K\gamma^2,$$
$$C_2\,[(g)] = Me^2 + N\gamma^2,$$

where E, F, G, H, K, M, N are functions of m, e' only, then

$$\frac{H}{K} = \frac{E}{F}, \quad \frac{M}{N} = \frac{F}{G}.$$

The proof of this theorem may be found in Adams' paper already referred to (IV). It is based on the consideration of the orders of certain functions. In the extension of this theorem which follows, I shall prove again Adams' theorem by another method which, though lacking altogether the neatness of his demonstration, has the advantage of putting the connections between the constant part of the Parallax of the Moon and the motions of the Perigee and Node in a rather clearer light.

Extension of Adams' Second Theorem.

Let

$$e^2T_e = 2\,\frac{a_0^2}{\varkappa}\Sigma_j\left[\{2j + 1 + m + (c)\}\,(\varepsilon)_j^2 + \{2j - 1 - m + (c)\}(\varepsilon')_{-j}^2\right],$$

$$\gamma^2T_\gamma = 4\,\frac{a_0^2}{\varkappa}\,\Sigma_j\,[2j + (g)](K)_j^2$$

(see equations (17) and (27)), *where T_e, T_γ are expressed in terms of the same eccentricity and latitude constants (which we have called e, γ) as $1/R$, (c), (g); then*

$$\left.\begin{aligned} HT_e &= 6E, & MT_\gamma &= 6F, \\ KT_e &= 6F, & NT_\gamma &= 6G, \end{aligned}\right\} \tag{41}$$

and thence

$$\frac{T_e}{T_\gamma} = \frac{M}{K}.$$

As before we omit the brackets which denote the presence of powers of e', for the sake of simplicity of expression: it is understood that they exist in what follows.

We have from equations (17) and (31) in this notation,

$$e^4 HT_e = C_{e^4}\left[\frac{x_e X_{e^2} + y_e Y_{e^2}}{R_{e^2}}\right],$$

$$e^2\gamma^2 KT_e = C_{e^2\gamma^2}\left[\frac{x_e(X_e + x_{\gamma^2}) + y_e(Y_e + y_{\gamma^2})}{\{(X_e + x_{\gamma^2})^2 + (Y_e + y_{\gamma^2})^2 + (z_\gamma + z_{e\gamma})^2\}^{\frac{3}{2}}}\right].$$

These can be included in the equation

$$e^4 HT_e + e^2\gamma^2 KT_e = C_{e^4,\ e^2\gamma^2}\left[\frac{x_1 X_2 + y_1 Y_2}{(R_2^2 + 2z_1 z_2)^{\frac{3}{2}}}\right]. \qquad (42)$$

For since x, y contain only even powers of γ, $x_1 = x_e$, $y_1 = y_e$, and when $\gamma = 0$, $X_2 = X_{e^2}$, etc., it therefore includes the first equation. If we want the second equation we neglect powers of e in X, Y above the first, hence $X_2 = X_e + x_{\gamma^2}$, etc., for this case.

Again we have from equations (30) and (27)

$$e^2\gamma^2 MT_\gamma = C_{e^2\gamma^2}\left[\frac{z_\gamma(z_\gamma + z_{e\gamma})}{R_{e^2}^3}\right],$$

$$\gamma^4 NT_\gamma = C_{\gamma^4}\left[\frac{z_\gamma^2}{(R_{\gamma^2}^2 + z_\gamma^2)^{\frac{3}{2}}}\right],$$

and these can be likewise included in the equation

$$e^2\gamma^2 MT_\gamma + \gamma^4 NT_\gamma = C_{e^2\gamma^2,\ \gamma^4}\left[\frac{z_1 Z_2}{R_2^3}\right], \qquad (43)$$

since z contains only odd powers of γ.

Now, since constant terms contain only even powers of e, γ, we have by addition of (42) and (43),

$$e^4 HT_e + e^2\gamma^2(KT_e + MT_\gamma) + \gamma^4 NT_\gamma = C_4\left[\frac{x_1 X_2 + y_1 Y_2 + z_1 Z_2}{(R_2^2 + 2z_1 z_2)^{\frac{3}{2}}}\right]. \qquad (44)$$

By means of this equation we can reduce the four equations (41) to two, namely,

$$C_4\left[\frac{6}{R_4}\right] = C_4\left[\frac{x_1 X_2 + y_1 Y_2 + z_1 Z_2}{(R_2^2 + 2z_1 z_2)^{\frac{3}{2}}}\right], \qquad (A)$$

$$C_{e^2\gamma^2}\left[\frac{x_1 X_2 + y_1 Y_2}{(R_2^2 + 2z_1 z_2)^{\frac{3}{2}}}\right] = C_{e^2\gamma^2}\left[\frac{z_1 Z_2}{R_2^3}\right]. \qquad (B)$$

For, suppose (A) and (B) to be proved. In (A) put $\gamma = 0$, then $z = 0$, and by

(44) and (40) the first of the relations (41) holds. Next, suppose $e=0$, then in a similar manner the last of relations (41) holds. Finally, consider the terms in (A) of the order $e^2\gamma^2$. Owing to (B), the second and third of equations (41) will hold. Hence all that is now necessary is to prove the two equations (A) and (B). Since the methods of proof may be of value in investigating similar relations for higher powers of e, γ in the parallax than the fourth, they are given separate statements.

14. *To prove that*

$$C_4\left[\frac{6}{R_4}\right]=C_4\left[\frac{x_1X_2+y_1Y_2+z_1Z_2}{(R_2^2+2z_1z_2)^{\frac{3}{2}}}\right]. \tag{A}$$

Attach to x_1, x_2, y_1, etc., a factor α with an index of the same order in e, γ as the quantities to which they are attached. Thus for x_1 we put αx_1, for y_2, $\alpha^2 y_2$, etc.; α will be omitted where its presence is unnecessary. Let

$$S_{rs}=x_rx_s+y_ry_s.$$

Then $$R_2^2=r_0^2+\alpha^2S_{11}+\alpha^4S_{22}+2\alpha S_{01}+2\alpha^2S_{02}+2\alpha^3S_{12}+\alpha^2z_1^2.$$

Hence, reducing to a common denominator, the expression

$$\begin{aligned}\frac{6}{R_2}-2\,\frac{x_0X_2+y_0Y_2}{R_2^3}&=\frac{2}{R_2^3}\,(2r_0^2+3\alpha^2S_{11}+3\alpha^2z_1^2+3\alpha^4S_{22}+5\alpha S_{01}+5\alpha^2S_{02}+6\alpha^3S_{12})\\&=\frac{1}{R_2^3}\,(4r_0^2+5\alpha^2S_{11}+5\alpha^2z_1^2+6\alpha^4S_{22}+9\alpha S_{01}+10\alpha^2S_{02}+11\alpha^3S_{12})\\&\qquad+\frac{x_1X_2+y_1Y_2+z_1^2}{R_2^3}\\&=\alpha^5.\,\frac{4\alpha^7r_0^2+5\alpha^9S_{11}+5\alpha^9z_1^2+6\alpha^{11}S_{22}+9\alpha^8S_{01}+10\alpha^9S_{02}+11\alpha^{10}S_{12}}{(\alpha^8r_0^2+\alpha^{10}S_{11}+\alpha^{10}z_1^2+\alpha^{12}S_{22}+2\alpha^9S_{01}+2\alpha^{10}S_{02}+2\alpha^{11}S_{12})^{\frac{3}{2}}}\\&\qquad+\frac{x_1X_2+y_1Y_2+z_1^2}{R_2^3}\\&=\alpha^5\,\frac{d}{d\alpha}\left(\frac{1}{R\alpha^4}\right)+\frac{x_1X_2+y_1Y_2+z_1^2}{R_2^3}.\end{aligned}$$

Now, owing to the definition of α we can consider $1/R_2$ as expansible in powers of α, and it is easily seen that

$$\alpha^5\,\frac{d}{d\alpha}\left(\frac{1}{R_2\alpha^4}\right)$$

contains no term of the fourth order in α. Hence

$$C_4\left[\frac{6}{R_2} - 2\frac{x_0X_2 + y_0Y_2}{R_2^3}\right] = C_4\left[\frac{x_1X_2 + y_1Y_2 + z_1^2}{R_2^3}\right]. \tag{45}$$

Let now $q = 2$ in the theorem of Art. 11. Equation (33) becomes

$$C_4\left[\frac{3}{R_4}\right] = C_4\left[\frac{3}{R_2} - \frac{x_0X_2 + y_0Y_2}{R_2^3} - 3\frac{z_1z_2S_{01}}{r_0^5}\right].$$

Hence from (45),

$$C_4\left[\frac{6}{R_4}\right] = C_4\left[\frac{x_1X_2 + y_1Y_2 + z_1^2}{R_2^3} - 6\frac{z_1z_2S_{01}}{r_0^5}\right]$$

$$= C_4\left[\frac{x_1X_2 + y_1Y_2 + z_1Z_2}{(R_2^2 + 2z_1z_2)^{\frac{3}{2}}}\right],$$

which is the result required.

The proof of equation (B) will be divided into two parts, of which the first contains a method of general application to the terms which involve both the constants e, γ.

15. *Lemma. To prove that*

$$C_{e^2\gamma^2}\left[\frac{x_{\gamma^2}X_{e^2} + y_{\gamma^2}Y_{e^2}}{R_{e^2}} - \frac{x_{e^2}X_{\gamma^2} + y_{e^2}Y_{\gamma^2}}{R_{\gamma^2}}\right] = 0. \tag{46}$$

Let, for the purposes of this proof only,

$$\begin{aligned} \xi &= x_0 + x_e + x_{e^2}, & \xi' &= x_0 + x_{\gamma^2} \quad , \\ \eta &= y_0 + y_e + y_{e^2}, & \eta' &= y_0 + y_{\gamma^2} \quad , \\ r^2 &= \xi^2 + \eta^2 \quad , & \zeta' &= z_\gamma \quad , \\ & & r'^2 &= \xi'^2 + \eta'^2 + \zeta'^2. \end{aligned}$$

Then ξ, η contain the terms in x, y, independent of γ, as far as the order e^2, and ξ', η', ζ', r', the terms in x, y, z, r, independent of e, as far as the order γ^2.

The proof of the lemma is not difficult. We proceed as in section 11. Taking the equations of motion (34) and substituting therein for x, y, z first ξ, η, 0 and then ξ', η', ζ', we have

$$\ddot{\xi} - 2n'\dot{\eta} - 3n'^2\xi + A'\xi + B'\eta = -\mu\frac{\xi}{r^3},$$

$$\ddot{\eta} + 2n'\dot{\xi} \qquad + B'\xi + C'\eta = -\mu\frac{\eta}{r^3},$$

and

$$\ddot{\xi}' - 2n'\dot{\eta}' - 3n'^2\xi' + A'\xi' + B'\eta' = -\mu\frac{\xi'}{r'^3},$$

$$\ddot{\eta}' + 2n'\dot{\xi}' \qquad\qquad + B'\xi' + C'\eta' = -\mu\frac{\eta'}{r'^3};$$

the right-hand sides of the first pair being expanded as far as the order e^2 and those of the second pair as far as the order γ^2. Take the sum of the first pair multiplied respectively by ξ', η' from the sum of the second pair multiplied respectively by ξ, η. The terms on the left-hand side of the resulting equation form a perfect differential, and therefore contain no constant part. Hence those on the right-hand side as far as the orders e^2, γ^2 contain no constant part. We have then in particular

$$C_{e^2\gamma^2}\left[(\xi\xi' + \eta\eta')\left(\frac{1}{r^3} - \frac{1}{r'^3}\right)\right] = 0.$$

Since ρ contains no terms in γ^2 and ρ' contains no terms in e^2, we have

$$C_{e^2\gamma^2}\left[\frac{\xi\xi' + \eta\eta'}{r^3} - \frac{\xi\xi' + \eta\eta'}{r'^3}\right] = 0,$$

whence follows immediately the equation (46).

From this result we can deduce the result (B). This seems to be most quickly proved by simple expansion.

16. *To prove that*

$$C_{e^2\gamma^2}\left[\frac{x_1X_2 + y_1Y_2}{(R_2^2 + 2z_1z_2)^{\frac{3}{2}}}\right] = C_{e^2\gamma^2}\left[\frac{z_1Z_2}{R_2^3}\right].$$

Let

$$S_{01} = x_0x_e + y_0y_e, \qquad S_{02'} = x_0x_{\gamma^2} + y_0y_{\gamma^2},$$

$$S_{02} = x_0x_{e^2} + y_0y_{e^2}, \qquad S_{12'} = x_ex_{\gamma^2} + y_ey_{\gamma^2},$$

$$S_{11} = x_e^2 + y_e^2, \qquad S_{22'} = x_{e^2}x_{\gamma^2} + y_{e^2}y_{\gamma^2}.$$

The orders of any terms will then be denoted by the suffixes of S. We have, with this notation, to the order required,

$$R_{e^2}^2 = r_0^2 + S_{11} + 2S_{01} + 2S_{02},$$

$$R_{\gamma^2}^2 = r_0^2 + 2S_{02'} + z_1^2,$$

and therefore, by expansion of equation (46),

$$C_{e^2\gamma^2}\left[\frac{S_{22'}}{r_0^3} - 3\frac{S_{01}S_{12'}}{r_0^5} + S_{02'}\left\{-3\frac{S_{02} + \frac{1}{2}S_{11}}{r_0^5} + \frac{15}{2}\frac{S_{01}^2}{r_0^7}\right\}\right.$$

$$\left. - \frac{S_{22'}}{r_0^3} + 3S_{02}\frac{S_{02'} + \frac{1}{2}z_1^2}{r_0^5}\right] = 0,$$

or, after reduction and multiplication by 2,

$$C_{e^2\gamma^2}\left[-6\,\frac{S_{01}\,S_{12'}}{r_0^5}-3\,\frac{S_{02'}\,S_{11}}{r_0^5}+15\,\frac{S_{01}^2\,S_{02'}}{r_0^7}+3\,\frac{S_{02}\,z_1^2}{r_0^5}\right]=0.$$

Rearranging, this equation may be written

$$\begin{aligned}C_{e^2\gamma^2}\Bigg[S_{01}&\left\{-3\,\frac{S_{12'}}{r_0^5}+15\,\frac{S_{01}\,S_{02'}+\frac{1}{2}z_1^2\,S_{01}}{r_0^7}\right\}\\&-3S_{11}\,\frac{S_{02'}+\frac{1}{2}z_1^2}{r_0^5}-3S_{12'}\,\frac{S_{01}}{r_0^5}-z_1^2\left\{-3\,\frac{S_{02}+\frac{1}{2}S_{11}}{r_0^5}+\frac{15}{2}\,\frac{S_{01}^2}{r_0^7}\right\}\Bigg]=0,\end{aligned}$$

which is the expansion of

$$C_{e^2\gamma^2}\left[\frac{S_{01}+S_{11}+S_{12'}-z_1^2}{(r_0^2+S_{11}+2S_{01}+2S_{02}+2S_{02'}+2S_{12'}+z_1^2)^{\frac{3}{2}}}\right]=0.$$

Referring to the notation defined by equations (32), this is evidently the same, if we omit superfluous terms, as

$$C_{e^2\gamma^2}\left[\frac{x_1X_2+y_1Y_2-z_1^2}{R_2^3}\right]=0.$$

This equation is easily shown to be the same as

$$C_{e^2\gamma^2}\left[\frac{x_1X_2+y_1Y_2-z_1Z_2}{(R_2^2+2z_1\,z_2)^{\frac{3}{2}}}\right]=0,$$

whence follows the equation (B).

The above results give some insight into the nature of the connection between the constant parts of the Moon's Parallax and the motions of its Perigee and Node. If we had assumed the results of Adams' second theorem (which is here a deduction from the equations (A) and (B)), it would have been unnecessary to prove (B). But it is desirable to see exactly by what means the results come, and thence to get an idea as to the way in which they might be extended to higher powers of e, γ. The methods followed above appear to be sufficient to indicate further relations if they exist. The chief trouble is to investigate the terms depending on both e, γ, those depending on one only of these constants requiring much less labor. The results are independent of the particular method of defining e, γ, so long as these quantities are of the same order as the coefficients of the principal elliptic term in longitude and of the principal term in latitude respectively. In fact, their definition is restricted only by the assumption that the coordinates are expressible by means of the same angles at any stage of the approximations. We simply approximate to the coefficients. A reference to Adams' paper (IV) will make these points clearer.

17. *Extension to Terms of Higher Orders.*

It is not difficult to extend these formulæ when terms of higher orders in the expressions of c, g are under consideration. But simple relations with the constant part of the Parallax of the Moon, like those which exist for $q = 1$ and $q = 2$, do not seem to be present in the terms of higher orders. The methods used here should give any relations which may exist, and it is not at all improbable that there are such.

The connections between the motions of Perigee and Node and the Parallax are really due to the fact that these are all independent (in the sense of the theorem of section 11) of the coefficients of the next higher order than the terms under consideration in the Perigee and Node. The formulæ for the latter, though symmetrical in x, y or z, are not so with reference to the quantities x_0, x_e, x_{γ^2}, etc., while the expression for the Parallax is so. Consequently the results of Adams' theorems and their extensions must be rather considered from this point of view as accidental.

The terms in the motion of the Perigee depend principally on the function

$$\frac{x_1 X_{2q-2} + y_1 Y_{2q-2}}{(R^2_{2q-2} + 2z_1 z_{2q-2})^{\frac{3}{2}}},$$

where $2q - 2$ is the order of the terms required.

As an example, take the case $q = 3$ in the theorem of Art. 11. This gives, if we consider only the constant part of the order e^6, i. e. putting $\gamma = 0$,

$$C_{e^6}\left[\frac{9}{R_6}\right] = C_{e^6}\left[\frac{9}{R_4} - 3\,\frac{x_0 X_4 + y_0 Y_4}{R_4^3}\right],$$

which may be shown to be equal to

$$C_{e^6}\left[\tfrac{1}{2}\alpha^7 \frac{d}{d\alpha}\left(\frac{1}{R\alpha^6}\right) + 2\,\frac{x_1 X_4 + y_1 Y_4}{R_4^3} + \frac{x_2 X_3 + y_2 Y_3}{R_4^3} - \frac{x_4 X_1 + y_4 Y_1}{R_4^3}\right],$$

after the method and notation used before. The first term on the right-hand side vanishes, having no coefficient of the order e^6; the second term is that required for the determination, in the motion of the Perigee, of the terms of order

e^4 [equation (18)], but the other two terms do not vanish. Some reductions are possible in these latter terms, but x_4 and y_4 cannot be eliminated.

PART III.

TERMS OF LONG AND SHORT PERIODS WHOSE COEFFICIENTS ARE LARGE IN COMPARISON TO THE ORDERS OF THE COEFFICIENTS.

There are some results which can be deduced from the infinite determinant which are of importance when short-period terms are under discussion.

Let the arguments of any series of terms in x, y differing by multiples of $2\nu(t-t_0)$ be

$$(2j+\Lambda)\nu t + \text{const.},$$

where Λ is of the form $j_1+km+pc+2qg$, and consequently the coefficients of the order $a'^{-j'}e'^{k'}e^{p'}\gamma^{2q'}$. Here k, p, q are integers positive or negative. Also $j_1=0$ when j' is even and $j_1=1$ when j' is odd; in addition, we have

$$k'=(k) \text{ or } k'=(k)+\text{even positive integer, etc.}$$

Let the coefficients corresponding to any given value of Λ be λ_j, λ'_j. The equations which determine them are

$$(2j+1+m+\Lambda)^2\lambda_j+\Sigma_i M_i\lambda_{j-i}+\Sigma_i N_i\lambda'_{i-j} = \text{known terms independent of } \lambda, \lambda',$$
$$(2j-1-m+\Lambda)^2\lambda'_{-j}+\Sigma_i M_{-i}\lambda'_{i-j}+\Sigma_i N_{-i}\lambda'_{j-i} = \quad \text{“} \quad \text{“} \quad \text{“} \quad \text{“}$$

Suppose that the solution of these equations is found by means of determinants. A reference to equations (14) shows that the common denominator in the expressions of all the coefficients λ_j, λ'_j is $\Delta(\Lambda)$ or $\nabla(\Lambda)$. Now the identity

$$\nabla(c) \equiv (\cos\pi c-\cos\pi c_1)(\cos\pi c-1)$$

of Art. 4 holds for all values of c, and therefore when we put Λ for c. Hence, dropping the suffix of c_1 so that c denotes the principal part of the motion of Perigee, we have for the common denominator $\nabla(\Lambda)$ the value

$$(\cos\pi\Lambda-\cos\pi c)(\cos\pi\Lambda-1).$$

From this we conclude that the coefficients λ_j, λ'_j may become large when either

$$\cos\pi\Lambda-\cos\pi c \quad \text{or} \quad \cos\pi\Lambda-1$$

is a small quantity. Since c is nearly equal to unity, these expressions cannot become small together. This shows that when we have the equations

$$\Lambda = c + \text{even integer} \quad \text{or} \quad \Lambda = \text{even integer},$$

approximately satisfied, the quantities λ_j, λ'_j may become large. The term "even integer" includes negative as well as positive even integers and also zero.

The second case is that of long-period terms. The result obtained does not add anything to our knowledge of them, except perhaps the fact (which applies also to the terms affected by the first equation) that all terms whose arguments differ from $\Lambda \nu t +$ const. by even multiples of $2\nu(t - t_0)$ should be considered together. The first equation affects only short-period terms: the periods of these terms differ little from those of the principal elliptic inequalities, that is, from $2\pi/(2j \pm c)\nu$. We have neglected in c all terms but those depending on m only, but the results evidently hold when we consider the full value of c. If we had used numerical values for all the constants it would have been necessary to take this full value. In the case before us, where we use the numerical value of m only, we take the part of c depending on m only.

It seems to have been usual to consider short-period terms as those whose periods approximate to that of the mean motion. The result shows that we ought rather to consider the terms whose periods approximate to that of the principal elliptic inequality. The result obtained in a previous investigation (VII, p. 254) is evidently only a first approximation to that obtained here. The difference is due to the fact that here we include at once all the terms whose arguments differ from that considered by even multiples of $2\nu(t - t_0)$, and we therefore include in the coefficients all powers of m.

Only a finite number of the coefficients λ_j, λ'_j will become large relatively to the rest, but the whole series will be large relatively to their order in e, e', γ, $1/a'$. No coefficients of terms of long period become large by the integration of the latitude equation. The same remarks, as to the terms of short period, apply if for "principal elliptic inequalities" we substitute "principal latitude inequalities," i. e. those of period $2\pi/(2j \pm g)\nu$.

For the calculations of Part I, where we are especially concerned, in the infinite determinant, with the values of c, g which depend on m only, the above remarks may be included in the following rule: *When calculating any series of terms in* x, y *in which* Λ *has the value* $i + km + pc + 2qg$, *in order to discover if a large divisor is present it is necessary to examine whether for positive or negative* *integral values of* i, k, p, q *either of the expressions*

$$i + km + (p - 1)c + 2qg, \quad i + km + pc + 2qg$$

becomes small. In calculating terms in z, *we examine whether the second expression may become small.* The last statement arises from the fact that in latitude Λ is of the form

$$i + km + pc + (2q + 1)g,$$

and this for large coefficients must be approximately equal to g. It is to be remembered that the rule applies to all terms wherein the coefficient of νt in the argument differs from Λ by even multiples of ν.

The part of the coefficient of lowest order of a term in x, y whose argument is $(i + km + pc + 2qg)\nu t +$ const., is of order $e'^k e^p \gamma^{2q}$ or $e'^k e^p \gamma^{2q} . 1/a'$, according as i is even or odd: in z the argument being $\{i + km + pc + (2q + 1)\}\nu t +$ const., the orders are for i even or odd, $e'^k e^p \gamma^{2q+1}$ and $e'^k e^p \gamma^{2q+1} 1/a'$ respectively. Hence *if there is a long-period inequality with a large coefficient of order* $e'^k e^p \gamma^{2q}$ *in* x, y *there is a short-period inequality with a large coefficient of order* $e'^k e^{p+1} \gamma^{2q}$ *in* x, y *and a similar one of order* $e'^k e^p \gamma^{2q+1}$ *in* z. In the indices of e, e', γ the numbers k, p, q receive positive values.

By considering the algebraical values of the principal parts of c, g we can also arrive at the order of the denominator in relation to m, and thence, by looking at the manner in which the denominators are built up, the orders of any terms, with respect to the parameter m, can be obtained.

Christ's College, Cambridge, *December 24th*, 1894.